Poloidal rotation and its relation to the potential vorticity flux

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A kinetic generalization of a Taylor identity appropriate to a strongly magnetized plasma is derived. This relation provides an explicit link between the radial mixing of a four–dimensional (4D) gyrocenter fluid and the poloidal Reynolds stress. This kinetic analog of a Taylor identity is subsequently utilized to link the turbulent transport of poloidal momentum to the mixing of potential vorticity. A quasilinear calculation of the flux of potential vorticity is carried out, yielding diffusive, turbulent equipartition, and thermoelectric convective components. Self-consistency is enforced via the quasineutrality relation, revealing that for the case of a stationary small amplitude wave population, deviations from neoclassical predictions of poloidal rotation can be closely linked to the growth/damping profiles of the underlying drift wave microturbulence. © *2010 American Institute of Physics*. [doi:10.1063/1.3490253]

I. INTRODUCTION

Poloidal and toroidal flows play a critical role in the regulation and stabilization of numerous modes within tokamak devices which are known to have detrimental effects on plasma confinement. Poloidal flows in particular have garnered significant attention as a result of the prominent role played by these flows within the radial force balance equation in the vicinity of edge or internal transport barriers,[1–3] as well as providing a potential trigger mechanism for barrier formation.[4] In spite of the critical role often attributed to these flows, an understanding of the physical processes underlying their generation remains elusive. While it is often assumed that the lack of poloidal symmetry within a tokamak device will result in the plasma poloidal flow being dragged back to its neoclassical level, numerous counterexamples in the vicinity of internal transport barriers can be identified within the experimental literature.[5–7] Furthermore, recent experiments at DIII-D indicate that deviations in the magnitude as well as direction of the poloidal rotation of impurity ions from that predicted by neoclassical theory can be present throughout the plasma volume.[8] Within the above studies, measured poloidal flows have been observed to deviate from neoclassical predictions by as much as an order of magnitude. These substantial deviations of poloidal flows from neoclassical predictions are suggestive of the critical role in which turbulent stresses are likely playing in the generation of mean poloidal flows. Further motivation for the theoretical study of turbulent stresses is provided by basic experiments which have unambiguously demonstrated the robust nature of turbulent flow generation in strongly magnetized plasmas.[9,10]

While existing formulations within the theoretical literature have derived criteria for turbulent poloidal flow generation,[11] the existing theoretical framework is not sufficient for determining regimes in which turbulent stresses are strong enough to driving experimentally relevant deviations from neoclassical predictions of poloidal rotation. Recent progress has been made in this regard via nonlinear full-f gyrokinetic simulations on the GYSELA code.[12] These simulations have shown that a significant component of the poloidal rotation can be driven by turbulent stresses in low collisionality regimes.[13] This result is particularly pressing since the next generation of confinement devices will likely operate in significantly lower collisionality regimes than current machines.

The combination of recent experimental and numerical findings suggest that mechanisms for driving poloidal rotation outside the scope of conventional neoclassical theories will likely be active in future devices. Within this analysis we seek to provide a theoretical framework for describing the underlying physical mechanisms through which plasma microturbulence may drive mean poloidal flows. Our motivation throughout this analysis is to build upon fundamental concepts originally developed within the context of highly idealized fluid models which lack much of the cumbersome technical complexity present within more comprehensive models of plasma turbulence. In particular, the study of wave-mean flow interactions within the context of atmospheric and oceanic systems has led to the development of an elegant set of theoretical tools for describing turbulent flow generation.[14–16] As a specific example, an expression for mean zonal momentum can be written as

$$\frac{\partial \overline{u_y}}{\partial t}+\frac{\partial}{\partial x}\overline{\delta u_y \delta u_x}=-\nu \overline{u_y}. \tag{1}$$

Here the notation is consistent with plasma physics conventions: $\hat{e}_x$ is the direction of inhomogeneity, $\hat{e}_y$ is the zonal direction, and the velocity perturbations can be written in

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terms of the electrostatic potential as $\mathbf{u}\equiv(c/B)(\hat{b}\times\nabla_\perp\phi)$. It is often useful to work with the flux of potential vorticity (PV) rather than the divergence of the Reynolds stress. This can be conveniently accomplished via the use of a Taylor identity,[17] which can be written as (in dimensionless notation)

$$\frac{\partial}{\partial x}\overline{\delta u_y \delta u_x} = -\overline{\delta q\,\delta u_x}, \tag{2}$$

where $\delta q\equiv(1-\nabla_\perp^2)\delta\phi$ is the PV, we have normalized length and time scales by ρ_s and ω_{ci}, respectively, the electrostatic potential by e/T_e, and $\overline{(\cdots)}$ corresponds to an average over the zonal coordinate. Equation (2) provides an explicit link between the mixing of PV and the Reynolds stress. Note that while we have assumed translational invariance in the zonal direction, no small amplitude assumption of the underlying fluctuations was necessary in the derivation of Eq. (2). Hence it is appropriate to the description of both eddy and wave turbulence, and thus corresponds to a useful relation for both theoretical as well as experimental studies of momentum transport.[18] If for definiteness, we now assume the limit of small amplitude drift waves and adiabatic electrons, the linearized Hasegawa–Mima equation[19] provides a convenient description for the evolution of potential vorticity, namely,

$$\frac{\partial\delta q}{\partial t} = -\delta u_x\frac{\partial\bar{q}}{\partial x} - \overline{u_y}\frac{\partial\delta q}{\partial y} + F[\delta q] - D[\delta q], \tag{3}$$

where the radial gradient of mean PV is given by

$$\frac{\partial\bar{q}}{\partial x} = \frac{\partial\ln n_0}{\partial x} - \frac{\partial^2\overline{u_y}}{\partial x^2}.$$

Here $F[\delta q]$ and $D[\delta q]$ correspond to forcing and dissipation, respectively. An expression for the Reynolds stress can be derived by solving Eq. (3) for δu_x, and substituting the result into Eq. (2), yielding (see Ref. 20 for a more in depth discussion related to the Hasegawa–Wakatani equation retaining all nonlinear terms)

$$\frac{\partial}{\partial x}\overline{\delta u_y \delta u_x} = \frac{1}{2}\frac{1}{\bar{q}'}\frac{\partial\overline{\delta q^2}}{\partial t} - [F(x)-D(x)]\frac{\overline{\delta q^2}}{\bar{q}'}, \tag{4}$$

where for simplicity we have assumed the forcing and dissipation to have the form $F[\delta q]=F(x)\delta q$ and $D[\delta q]=D(x)\delta q$. Equation (4), while resulting from an elementary set of operations, can be seen to put a strong constraint on the generation of mean flows. Namely, from Eq. (4) it is clear that for the small amplitude limit considered here, the turbulent stress must vanish in the limit of conservative stationary waves. Hence, Eq. (4) can be understood to correspond to a nonacceleration theorem for the mean flow. To understand the physical origin of the first term on the right-hand side (RHS) of Eq. (4) in more detail it is useful to note that consistent with the small amplitude assumption utilized above we may write

$$-\frac{1}{2}\frac{\overline{\delta q^2}}{\bar{q}'} = \sum_k k_y\frac{E_k}{\omega_k}, \tag{5}$$

where E_k is the energy density and ω_k is the linear frequency. For media which are translationally invariant in the zonal direction, it can be readily shown that the quantity on the RHS of Eq. (5) is conserved up to dissipative losses in the small amplitude limit,[21,22] and is often referred to as a wave momentum density. Within this interpretation, the first term in Eq. (4) can be seen to represent the rate of change of the wave momentum density. Hence, for conservative systems, Eq. (4) indicates that any increase or decrease in the wave momentum density must necessarily drive a finite Reynolds stress.

Considering the opposite limit of a nonconservative stationary (small amplitude) wave population it is clear that the Reynolds stress is explicitly linked to the forcing and dissipation profiles of the underlying modes. It is useful to note that in order for a stationary state to exist, the forcing and dissipation profiles must necessarily satisfy the constraint $\int d^2x[F(x)-D(x)]\delta q^2=0$. Thus, excluding trivial solutions, this self-consistency constraint necessitates the coexistence of regions above and below marginality, or equivalently, regions of emission and absorption of wave momentum. As indicated by Eq. (4), regions of emission $[F(x)>D(x)]$ must drive a positive stress (for $\bar{q}'<0$), whereas regions of absorption $[F(x)<D(x)]$ will drive a negative stress. An explicit expression for the mean flow profile can be derived by substituting Eq. (4) into Eq. (1), yielding (at stationarity)

$$\overline{u_y} = -\frac{2}{\nu}[F(x)-D(x)]\sum_k k_y\frac{E_k}{\omega_k}. \tag{6}$$

From Eq. (6) it is clear that within the simple limit considered here, the equilibrium flow profile is dependent upon the dissipation and drive of the underlying fluctuations. We note in passing that this simple form for the flow provides an attractive means of determining the stability of the mean flow profile. Namely, as discussed in Ref. 23, expressions of this form may be used to determine whether a Rayleigh–Kuo inflection-point is present within the flow profile, a necessary condition for Kelvin–Helmholtz instability.[24]

Our focus within this work is on the derivation of a theoretical framework which will allow for the implementation of many of the key concepts outlined above to more comprehensive models of plasma turbulence. Central to this study will be the identification of kinetic analogs to many of the fluid concepts discussed above such as potential vorticity and Taylor identities. As Eq. (6) makes explicit, the irreversible transport of momentum can be explicitly linked to the violation of conservative wave dynamics. Hence, within the following analysis we will strictly enforce self-consistency throughout, such that sources of irreversible momentum transport may be identified. The remainder of this paper is organized as follows. Section II contains a derivation of the transport equation for poloidal momentum utilized in this analysis. In Sec. III, the generalization of Eq. (2) to strongly magnetized plasmas is carried out. Section IV presents a quasilinear calculation of the potential vorticity flux in both the fluid and kinetic limits. In Sec. V, a discussion and conclusion is presented.

II. TRANSPORT EQUATIONS

Within this section we seek to derive a closed system of equations describing the plasma flow evolution which incorporates both neoclassical as well as turbulent stresses. Previous formulations of turbulent momentum transport have utilized parallel and radial force balance equations, and then closed the system via *ad hoc*, but physically reasonable, assumptions of the poloidal rotation. Here we seek to close the system self-consistently by introducing a third constraint, namely, toroidal force balance. Before proceeding further it is useful to express perpendicular force balance in the form[25,26]

$$u_\parallel = -c\left(\frac{\partial\phi}{\partial\psi} + \frac{1}{n_i e}\frac{\partial P_i}{\partial\psi}\right)\frac{I(\psi)}{B} + K(\psi)B, \tag{7}$$

where we have assumed the total mean flow to be incompressible, the equilibrium magnetic field is taken to be of the form

$$\mathbf{B} = I(\psi)\,\nabla\,\varphi + \nabla\varphi\times\nabla\psi,$$

and we note that $B_\varphi = I(\psi)/R$ where R is the major radius, ψ is the poloidal flux function, and the toroidal and poloidal directions are given by $\hat{e}_\varphi = R\nabla\varphi$ and $\hat{e}_\theta = (R/|\nabla\psi|)\nabla\varphi\times\nabla\psi$, respectively. This choice of representation for the equilibrium magnetic field, while not the most general,[27] will be particularly convenient for the axisymmetric system considered here. From Eq. (7) it is clear that we will need to determine expressions for the three independent variables $E_r \sim u_y^{(EB)}$ (where the binormal direction is defined by $\hat{e}_y \equiv \hat{b}\times\hat{e}_r$), $K(\psi) = u_\theta/B_\theta$, and $u_\parallel$. Note that here and throughout we will assume the pressure and density profiles to be known quantities.

After summing over particle species, and taking $m_e/m_i \to 0$, the parallel force balance equation is given by[28]

$$\frac{\partial}{\partial t}\langle\overline{n_i m_i B u_\parallel}\rangle + \langle\overline{\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}}\rangle = 0, \tag{8}$$

where $\langle\cdots\rangle$ and $(\overline{\cdots})$ correspond to flux surface and statistical averages, respectively. The stress tensor is defined by

$$\mathbf{\Pi} \equiv \sum_s m_s \int d^3 v f_s\left(\mathbf{vv} - \frac{1}{3}\mathbf{I}|\mathbf{v}-\mathbf{u}|^2\right),$$

and $\mathbf{I}$ is the identity matrix. The third constraint applied to the system will be toroidal force balance given by

$$\frac{\partial}{\partial t}\langle\overline{n_i m_i R u_\varphi}\rangle + \langle\overline{R\hat{e}_\varphi\cdot\nabla\cdot\mathbf{\Pi}}\rangle - \frac{e}{c}\langle\overline{\mathbf{j}\cdot\nabla\psi}\rangle = 0. \tag{9}$$

For axisymmetric systems this expression may be simplified by noting the identity[29]

$$\langle R\hat{e}_\varphi\cdot\nabla\cdot\mathbf{\Pi}\rangle = \langle\nabla\cdot(R\hat{e}_\varphi\cdot\mathbf{\Pi})\rangle.$$

Utilizing this identity and taking $\langle\mathbf{j}\cdot\nabla\psi\rangle = 0$ [an immediate consequence of the equilibrium condition $\mathbf{j}\times\mathbf{B}/c = \nabla p$ when $p = p(\psi)$], Eq. (9) can be written as

$$\frac{\partial}{\partial t}\langle\overline{n_i m_i R u_\varphi}\rangle + \langle\nabla\cdot(\overline{R\hat{e}_\varphi\cdot\mathbf{\Pi}})\rangle = 0. \tag{10}$$

Here it will be convenient to limit the analysis to stationary solutions and to separate the stress tensor into turbulent and collisional contributions, i.e., $\mathbf{\Pi} = \mathbf{\Pi}_{\rm turb} + \mathbf{\Pi}_{\rm neo}$ (see Appendix A for details). For the parallel direction, we estimate the viscous stress as follows:

$$\langle\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\rm neo}\rangle = -\frac{3}{2}\langle\Pi_{\parallel\parallel}^{(\rm neo)}\hat{b}\cdot\nabla B\rangle, \tag{11}$$

where $\Pi_{\parallel\parallel}^{(\rm neo)} = (2/3)(P_\parallel - P_\perp)$, and we have neglected off-diagonal contributions to $\mathbf{\Pi}_{\rm neo}$. Following the Hirshman–Sigmar method,[30] Eq. (11) can be approximated by

$$\langle\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\rm neo}\rangle \approx \frac{n_i m_i}{\tau_{ii}}\langle B^2\rangle[\mu_{00}K(\psi) + \mu_{01}L(\psi)], \tag{12}$$

where $K(\psi)$ and $L(\psi)$ are related to the poloidal flow of mass and heat, respectively, and μ_{00}, μ_{01} are "viscosity" coefficients whose values in various collisionality regimes can be found in Ref. 30, and τ_{ii} is the ion-ion collision time.

It is convenient to expand the turbulent contribution to the stress tensor as follows:

$$\begin{aligned}\langle\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\rm turb}\rangle &= \langle B_\varphi\hat{e}_\varphi\cdot\nabla\cdot\mathbf{\Pi}_{\rm turb}\rangle + \langle B_\theta\hat{e}_\theta\cdot\nabla\cdot\mathbf{\Pi}_{\rm turb}\rangle \\ &= \left\langle\frac{B_\varphi}{R}\nabla\cdot(R\hat{e}_\varphi\cdot\mathbf{\Pi}_{\rm turb})\right\rangle \\ &\quad + \left\langle\left(\frac{B_\theta}{r}\right)\nabla\cdot(r\hat{e}_\theta\cdot\mathbf{\Pi}_{\rm turb})\right\rangle \\ &\quad - \left\langle\left(\frac{B_\theta}{r}\right)\mathbf{\Pi}_{\rm turb}:\nabla(r\hat{e}_\theta)\right\rangle,\end{aligned} \tag{13}$$

where $\mathbf{A}:\mathbf{B} \equiv A_{jk}B_{kj}$. An expression for the poloidal stress can be derived by multiplying the toroidal force balance equation [Eq. (10)] by $I(\psi)/\langle R^2\rangle$ and subtracting it from the parallel force balance equation [Eq. (8)], yielding

$$\begin{aligned}0 = I(\psi)&\left\langle\left(\frac{1}{R^2} - \frac{1}{\langle R^2\rangle}\right)\nabla\cdot(\overline{R\hat{e}_\varphi\cdot\mathbf{\Pi}_{\rm turb}})\right\rangle \\ &+ \left\langle\left(\frac{B_\theta}{r}\right)\nabla\cdot(\overline{r\hat{e}_\theta\cdot\mathbf{\Pi}_{\rm turb}})\right\rangle - \left\langle\left(\frac{B_\theta}{r}\right)\overline{\mathbf{\Pi}_{\rm turb}:\nabla(r\hat{e}_\theta)}\right\rangle \\ &+ \frac{n_i m_i}{\tau_{ii}}\langle B^2\rangle[\mu_{00}K(\psi) + \mu_0 L(\psi)].\end{aligned} \tag{14}$$

It is apparent from Eq. (14) that both toroidal as well as poloidal stresses are capable of influencing the rate of poloidal rotation. While a direct evaluation of Eq. (14) is possible, it will be convenient to assume a simplified geometry in order to reduce the technical complexity of the analysis. Namely, we will assume a cylindrical equilibrium with a magnetic field of the form $B = B(r)$ for the remainder of this analysis. In this limit Eq. (14) reduces to

$$0 = \langle \nabla \cdot (\overline{r\hat{e}_\theta \cdot \mathbf{\Pi}_{\text{turb}}}) \rangle + n_i m_i r \mu_{ii}^{(\text{neo})}(u_\theta - u_\theta^{\text{neo}}), \tag{15}$$

where

$$\mu_{ii}^{(\text{neo})} \equiv \frac{1}{\tau_{ii}} \frac{\langle B^2 \rangle}{B_\theta^2} \mu_{00},$$

and we have used $u_\theta = K(\psi) B_\theta$ and defined $u_\theta^{\text{neo}} = -(\mu_{01}/\mu_{00}) L(\psi) B_\theta$. Here $L(\psi)$ is determined by the energy weighted momentum equation,[30] which is likely to also be impacted by turbulent stresses. The calculation of these additional stresses will be left for future work. Also, note that while formally the neoclassical friction vanishes in this simple limit, we will retain this contribution in order to be able to compare the strength of the turbulent poloidal stress versus the neoclassical friction.

III. DERIVATION OF KINETIC ANALOG OF TAYLOR IDENTITY

As discussed in Sec. II, we will be interested in computing the poloidal Reynolds stress given by the first term in Eq. (15). It will be convenient to separate this stress into its perpendicular and parallel components. For an axisymmetric system we have $\nabla\varphi \cdot \nabla\psi = \nabla\varphi \cdot \nabla\theta = 0$, which allows the vector relation to be straightforwardly derived,

$$\hat{e}_\theta = \frac{B_\varphi}{B} \hat{e}_y + \frac{B_\theta}{B} \hat{b}, \tag{16}$$

such that we may write the poloidal stress as

$$\Pi_{\theta r}^{(\text{turb})} = \frac{B_\varphi}{B} \Pi_{yr}^{(\text{turb})} + \frac{B_\theta}{B} \Pi_{\parallel r}^{(\text{turb})}, \tag{17}$$

where the perpendicular (binormal) direction is defined by $\hat{e}_y \equiv \hat{b} \times \hat{e}_r$. Explicit expressions for turbulent contributions to the perpendicular and parallel stresses are derived in Appendix A and are shown to be given by

$$\Pi_{yr}^{(\text{turb})} = m_i \frac{c}{B} \overline{\delta(n_i u_y)\, \delta E_y}, \tag{18a}$$

$$\Pi_{\parallel r}^{(\text{turb})} = m_i \frac{c}{B} [\overline{\delta(n_i u_\parallel)\, \delta E_y} + \overline{\delta(n_i u_y)\, \delta E_\parallel}], \tag{18b}$$

such that the poloidal stress may be written as

$$\Pi_{\theta r}^{(\text{turb})} = m_i \frac{c}{B} \left\{ \frac{B_\varphi}{B} \overline{\delta(n_i u_y)\, \delta E_y} + \frac{B_\theta}{B} [\overline{\delta(n_i u_\parallel)\, \delta E_y} + \overline{\delta(n_i u_y)\, \delta E_\parallel}] \right\} = m_i \frac{c}{B} \left\{ \overline{\delta(n_i u_y)\, \delta E_\theta} + \frac{B_\theta}{B} \overline{\delta(n_i u_\parallel)\, \delta E_y} \right\}. \tag{19}$$

Similarly, we note that the toroidal stress can also be seen to be composed of both parallel and perpendicular stresses, i.e.,

$$\Pi_{\varphi r}^{(\text{turb})} = m_i \frac{c}{B} \left\{ \frac{B_\varphi}{B} [\overline{\delta(n_i u_\parallel)\, \delta E_y} + \overline{\delta(n_i u_y)\, \delta E_\parallel}] - \frac{B_\theta}{B} \overline{\delta(n_i u_y)\, \delta E_y} \right\}. \tag{20}$$

Before proceeding further, it is useful to discuss the physical origin of the stresses contained in Eqs. (18a) and (18b). The perpendicular stress as well as the first contribution to the parallel stress can be recognized as $E \times B$ convection of perpendicular[11,31] and parallel momentum,[32–35] respectively. The second contribution to Eq. (18b), in contrast, can be seen to be necessary in order to ensure the symmetry of the stress tensor (i.e., $\Pi_{\parallel r} = \Pi_{r \parallel}$), and has been shown to be linked to polarization charge.[36,37] It is thus convenient to refer to this contribution as a parallel polarization stress. In order to further elucidate the physical origin of the perpendicular and parallel stresses, it is useful to consider the limit of small amplitude fluctuations. In this limit, a straightforward calculation allows the perpendicular stress to be written as

$$m_i \frac{c}{B} \overline{\delta(n_i u_y)\, \delta E_y} = \sum_k v_{gr} k_y N_k,$$

where $N_k \equiv E_k/\omega_k$ is the wave action density, E_k is the wave energy density, and v_{gr} is the radial group velocity. Thus within this simple limit, this contribution can be understood to correspond to a perpendicular wave stress. Similarly, the parallel polarization stress [second contribution in Eq. (18b)] can be shown to be linked to parallel wave stresses, i.e., an analogous calculation to that above yields

$$m_i \frac{c}{B} \overline{\delta(n_i u_y)\, \delta E_\parallel} = \sum_k v_{gr} k_\parallel N_k,$$

where we have again assumed the background fluctuations to be described by small amplitude fluctuations. Thus, both perpendicular and parallel wave stresses can be seen to influence mean flow evolution.

In order to derive a kinetic analog of the Taylor identity discussed above, it will be useful to consider the quantity given by

$$\Upsilon \equiv \overline{r \sum_s q_s \int d^3\bar{v}\, \delta F_s (\hat{e}_\varphi \times \nabla_\perp J_0 \delta\phi)_r}, \tag{21}$$

where $\int d^3\bar{v} \equiv 2\pi \int d\mu dv_\parallel B_\parallel^* \approx 2\pi \int d\mu dv_\parallel B$, $\mathbf{B}^* \equiv \mathbf{B} + v_\parallel \frac{m_i c}{e} \nabla \times \hat{b}$, and $B_\parallel^* \equiv \hat{b} \cdot \mathbf{B}^*$. Equation (21) may be reduced by a more involved, but analogous means as that utilized in the derivation of Eq. (2) above. Namely, it will be convenient to introduce a two-scale analysis whereby the spatial scales are separated into a set of "fast" variables associated with the rapidly varying microturbulence, and a set of "slow" variables associated with the variation of the mean fields. Here we will denote the fast variables by $\mathbf{x}_\perp$ and the slow variables by $\mathbf{X}_\perp$. This separation allows for spatial derivatives to be written in the form

$$\nabla_\perp \to \nabla_\perp^{(0)} + \varepsilon \nabla_\perp^{(1)}, \tag{22}$$

where $\varepsilon \sim \rho_i / L_\perp$, $L_\perp$ is a slow perpendicular length scale such as a density or temperature gradient, $\nabla_\perp^{(0)}$ is a derivative with respect to $\mathbf{x}_\perp$, and $\nabla_\perp^{(1)}$ corresponds to a derivative with respect to $\mathbf{X}_\perp$. Similarly, the fields will be ordered as

$$\phi = \phi^{(0)}(\mathbf{X}_\perp) + \varepsilon\,\delta\phi^{(1)}(\mathbf{x},\mathbf{X}_\perp,t) + \varepsilon^2\,\delta\phi^{(2)}(\mathbf{x},\mathbf{X}_\perp,t) + \cdots, \tag{23a}$$

$$F_s = F_s^{(0)}(\mathbf{X}_\perp) + \varepsilon\,\delta F_s^{(1)}(\mathbf{x},\mathbf{X}_\perp,t) + \varepsilon^2\,\delta F_s^{(2)}(\mathbf{x},\mathbf{X}_\perp,t) + \cdots, \tag{23b}$$

where for simplicity we will assume the microturbulence to be electrostatic. Also, an average over the fast variables may be defined such that $\overline{\delta\psi(\mathbf{x},\mathbf{X}_\perp,t)}=0$, but functions of only slow variables are left unaltered, i.e., $\overline{\psi(\mathbf{X}_\perp,t)}=\psi(\mathbf{X}_\perp,t)$. Furthermore, averages over the fast scales annihilate derivatives of fast variables as well as derivatives in the poloidal direction, but commute with slow derivatives in the radial direction, i.e., $\overline{\nabla_\perp^{(0)}(\cdots)}=\overline{\partial/\partial\theta(\cdots)}=0$, but $\overline{\hat{e}_r\cdot\nabla_\perp^{(1)}(\cdots)} =\hat{e}_r\cdot\nabla_\perp^{(1)}\overline{(\cdots)}$.

Before proceeding further it is convenient to note some properties of the operator $J_0(\lambda)$ since this quantity will appear frequently throughout the remainder of this section. This operator may be defined via its series representation, i.e.,

$$J_0(\lambda) = \sum_{m=0}^{\infty} \frac{(1/4)^m}{m!\Gamma(m+1)} |\rho_\perp \nabla_\perp|^{2m}. \tag{24}$$

Separating the perpendicular derivative into a fast and slow component, Eq. (24) can be written in the long wavelength limit as

$$J_0^{(0)}(\lambda) \approx 1 + \tfrac{1}{4}\rho_\perp^2 \nabla_\perp^{(0)^2}, \tag{25a}$$

$$J_0^{(1)}(\lambda) \approx \tfrac{1}{4}\rho_\perp^2 (\nabla_\perp^{(1)}\cdot\nabla_\perp^{(0)} + \nabla_\perp^{(0)}\cdot\nabla_\perp^{(1)}). \tag{25b}$$

Noting these definitions it is straightforward to show that

$$\overline{\delta f J_0^{(0)}(\lambda)\,\delta g} = \overline{\delta g J_0^{(0)}(\lambda)\,\delta f}, \tag{26}$$

whereas an analogous relation for $J_0^{(1)}$ cannot be derived without the introduction of surface terms.

A mildly simplified, but convenient form of the quasineutrality relation can be written in gyrocenter coordinates as[38,39]

$$\nabla_\perp\cdot\left(n_0 m_i \frac{c^2}{B^2}\nabla_\perp \delta\phi\right) = -\sum_s q_s \int d^3\bar{v}\, J_0(\lambda)\,\delta F_s. \tag{27}$$

Here we have taken the electron Debye length to zero, and for consistency we consider the long wavelength limit $k_\perp\rho_i < 1$, so that terms of order $\mathcal{O}(k_\perp^4\rho_i^4)$ may be neglected. It will be useful to utilize the multiscale framework described above in order to expand Eq. (21) order by order in ε. To lowest order, Eqs. (21) and (27) can be written as

$$\Upsilon^{(2)} = r\overline{\sum_s q_s \int d^3\bar{v}\,\delta F_s^{(1)}(\hat{e}_\varphi\times\nabla_\perp^{(0)} J_0^{(0)}\delta\phi^{(1)})_r} = -\overline{\sum_s q_s \int d^3\bar{v}\,\delta F_s^{(1)} J_0^{(0)} \frac{\partial^{(0)}\delta\phi^{(1)}}{\partial\theta^{(0)}}}, \tag{28a}$$

$$n_0 m_i \frac{c^2}{B^2}\nabla_\perp^{(0)^2}\delta\phi^{(1)} = -\sum_s q_s \int d^3\bar{v}\, J_0^{(0)}\delta F_s^{(1)}. \tag{28b}$$

Utilizing Eq. (26), and substituting Eq. (28b) into Eq. (28a) yields

$$\Upsilon^{(2)} = \overline{n_0 m_i \frac{c^2}{B^2}\nabla_\perp^{(0)^2}\delta\phi^{(1)}\frac{\partial^{(0)}}{\partial\theta^{(0)}}\delta\phi^{(1)}} = -\frac{1}{2}n_0 m_i\frac{c^2}{B^2}\overline{\frac{\partial^{(0)}}{\partial\theta^{(0)}}|\nabla_\perp^{(0)}\delta\phi^{(1)}|^2} = 0, \tag{29}$$

where we have used $B=B(r)$ and $n_0=n_0(r)$, a result of the simplified geometry being utilized possessing translational symmetry in the poloidal direction.

To next order in ε, Eq. (21) can be written as

$$\begin{aligned}\Upsilon^{(3)} = &-\overline{\sum_s q_s \int d^3\bar{v}\,\delta F_s^{(1)}\frac{\partial^{(0)}}{\partial\theta^{(0)}}J_0^{(0)}\delta\phi^{(2)}} -\overline{\sum_s q_s \int d^3\bar{v}\,\delta F_s^{(1)}\frac{\partial^{(0)}}{\partial\theta^{(0)}}J_0^{(1)}\delta\phi^{(1)}}\\ &-\overline{\sum_s q_s \int d^3\bar{v}\,\delta F_s^{(1)}\frac{\partial^{(1)}}{\partial\theta^{(1)}}J_0^{(0)}\delta\phi^{(1)}} -\overline{\sum_s q_s \int d^3\bar{v}\,\delta F_s^{(2)}\frac{\partial^{(0)}}{\partial\theta^{(0)}}J_0^{(0)}\delta\phi^{(1)}}.\end{aligned} \tag{30}$$

The third term in Eq. (30) can be shown to vanish identically, i.e.,

$$-\overline{\sum_s q_s \int d^3\bar{v}\,\delta F_s^{(1)}\frac{\partial^{(1)}}{\partial\theta^{(1)}}J_0^{(0)}\delta\phi^{(1)}} = \overline{n_0 m_i\frac{c^2}{B^2}\nabla_\perp^{(0)^2}\delta\phi^{(1)}\frac{\partial^{(1)}}{\partial\theta^{(1)}}\delta\phi^{(1)}} = -\frac{1}{2}n_0 m_i\frac{c^2}{B^2}\overline{\frac{\partial^{(1)}}{\partial\theta^{(1)}}|\nabla_\perp^{(0)}\delta\phi^{(1)}|^2} = 0,$$

where we have used Eqs. (26) and (28b). After rearranging the remaining terms, and again using Eqs. (26) and (28b), Eq. (30) can be written as

$$\Upsilon^{(3)} = -\overline{\delta q_{\mathrm{eff}}^{(2)}\frac{\partial^{(0)}}{\partial\theta^{(0)}}\delta\phi^{(1)}} - \overline{\sum_s q_s\int d^3\bar{v}\,\delta F_s^{(1)}\frac{\partial^{(0)}}{\partial\theta^{(0)}}J_0^{(1)}\delta\phi^{(1)}}, \tag{31a}$$

where

$$\delta q_{\mathrm{eff}}^{(2)} \equiv \sum_s q_s \int d^3\bar{v} J_0^{(0)} \delta F_s^{(2)} + e n_0 \rho_i^2 \nabla_\perp^{(0)^2} \frac{e\delta\phi^{(2)}}{T_{\perp i}}. \tag{31b}$$

Utilizing the definition given by Eq. (31b), after some manipulation, the gyrokinetic Poisson equation [Eq. (27)] can be written to second order as

$$\delta q_{\mathrm{eff}}^{(2)} = -\nabla_\perp^{(1)} \cdot \left[n_0 m_i \frac{c^2}{B^2} \nabla_\perp^{(0)} \delta\phi^{(1)} + \frac{1}{4}\sum_s q_s \int d^3\bar{v} \rho_\perp^2 \nabla_\perp^{(0)} \delta F_s^{(1)} \right] - \nabla_\perp^{(0)} \cdot \left[n_0 m_i \frac{c^2}{B^2} \nabla_\perp^{(1)} \delta\phi^{(1)} + \frac{1}{4}\sum_s q_s \int d^3\bar{v} \rho_\perp^2 \nabla_\perp^{(1)} \delta F_s^{(1)} \right]. \tag{32}$$

The first term in brackets can be recognized as the $E\times B$ velocity, whereas the second term corresponds to one half of the diamagnetic velocity. Substituting Eq. (32) into Eq. (31a) and combining terms yields

$$\Upsilon \approx \frac{1}{r}\frac{\partial}{\partial r} r^2 \overline{\left(n_0 m_i \frac{c}{B} \delta E_r - \sum_s \mathrm{sgn}(q_s) m_s \int d^3\bar{v} \frac{\mu B}{\omega_{cs}} \frac{\partial \delta F_s}{\partial r} \right) \frac{c}{B} \delta E_\theta}, \tag{33}$$

where we have dropped all superscripts in order to simplify the notation. The first term in parenthesis in Eq. (33) can be easily recognized as the $E\times B$ drift, whereas the second term corresponds to the diamagnetic drift. More explicitly, from Appendix B the ion perpendicular flow velocity can be written in gyrocenter coordinates as

$$\delta(n_i u_y) = \frac{1}{\omega_{ci}} \int d^3\bar{v} \mu B [J_0(\lambda) + J_2(\lambda)] \frac{\partial \delta F_i}{\partial r} - n_0 \frac{v_{\perp thi}^2}{\omega_{ci}} [\Gamma_1(b) - \Gamma_0(b)] \frac{\partial}{\partial r} \left(\frac{e\delta\phi}{T_{\perp i}} \right). \tag{34}$$

In the long wavelength limit this expression can be written as

$$\delta(n_i u_y) = \frac{1}{\omega_{ci}} \int d^3\bar{v} \mu B \frac{\partial \delta F_i}{\partial r} - n_0 \frac{c}{B} \delta E_r. \tag{35}$$

Thus, consistent with the approximations made above (i.e., long wavelength limit and $m_e/m_i \to 0$), we may rewrite Eq. (33) in the form

$$\frac{\Upsilon}{r} \equiv \overline{\sum_s q_s \int d^3\bar{v} \delta F_s (\hat{e}_\varphi \times \nabla_\perp J_0 \delta\phi)_r} \approx -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 m_i \frac{c}{B} \overline{\delta(n_i u_y) \delta E_\theta} \right]. \tag{36}$$

Equation (36) provides a kinetic analog of the Taylor identity given by Eq. (2). We note that during the derivation of Eq. (36) we have made explicit use of both the poloidal symmetry of the system as well as the mathematical structure of the polarization charge, i.e., the left-hand side (LHS) of Eq. (27) can be written in the form $\rho_{\mathrm{pol}} = \nabla \cdot \mathbf{P}$. The former of these properties is exactly satisfied within the cylindrical geometry utilized, and should be satisfied up to ballooning corrections for a tokamak plasma. The latter property is common to numerous gyrokinetic formulations (see Ref. 40). Hence, while the derivation of Eq. (36) was performed in a rather narrow limit, analogous relations are likely present within a diverse set of systems. As a specific example, a derivation of a relation analogous to Eq. (36) is performed in Appendix B in a homogeneous slab geometry incorporating full finite Larmor radius corrections.

IV. POTENTIAL VORTICITY FLUX

Within this section we focus on the evaluation of the gyrocenter flux in both the fluid and kinetic limits. In the former case, we will utilize a fairly simplified model in order to more easily identify parallels between particle and parallel momentum transport, as well as exploit familiar fluid concepts such as potential vorticity. For the latter case, which is likely appropriate for the description of low collisionality plasmas near marginality, we will implement a more comprehensive model to facilitate comparison with simulation and experiment. In both cases self-consistency will be enforced via the quasineutrality relation.

A. Fluid limit

In the previous section the poloidal stress divergence was linked to the flux of gyrocenter charge. Here we will be interested in evaluating this flux via quasilinear theory. Utilizing Eq. (19), Eq. (36) can be rewritten in terms of the turbulent poloidal stress as

$$\overline{\sum_s q_s \int d^3\bar{v} \delta F_s (\hat{e}_\varphi \times \nabla_\perp J_0 \delta\phi)_r} = -\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \left(\Pi_{\theta r}^{(\mathrm{turb})} - m_i \frac{c}{B}\frac{B_\theta}{B} \overline{\delta(n_i u_\|) \delta E_y} \right). \tag{37}$$

Substituting Eq. (37) into Eq. (15) yields an expression for the poloidal velocity,

$$-\overline{\sum_s q_s \int d^3\bar{v} \delta F_s (\hat{b} \times \nabla_\perp J_0 \delta\phi)_r} + n_i m_i \mu_{ii}^{(\mathrm{neo})} (u_\theta - u_\theta^{\mathrm{neo}}) = 0. \tag{38}$$

Here we have taken $B_\theta/B \ll 1$ such that we may approximate $\hat{e}_\varphi \approx \hat{b}$ and we have neglected the second term on the RHS of Eq. (37). After separating the ion gyrocenter and particle fluxes and Fourier transforming the former, Eq. (38) can be written as

$$n_i m_i \mu_{ii}^{(\mathrm{neo})}(u_\theta - u_\theta^{\mathrm{neo}}) = -ie\overline{\sum_k \int d^3\bar{v}(\hat{b}\times\mathbf{k}_\perp)_r J_0 \delta\phi_{-k}\delta F_{i,k}} - e\overline{(\hat{b}\times\nabla_\perp\delta\phi)_r\delta n_e}. \quad (39)$$

The gyrocenter flux can be approximated by the use of the linearized gyrokinetic equation given by[39]

$$\frac{\partial\delta F_i}{\partial t} + \dot{\mathbf{X}}^{(0)}\cdot\nabla\delta F_i + \dot{V}_\parallel^{(0)}\frac{\partial\delta F_i}{\partial v_\parallel} = -\dot{\mathbf{X}}^{(1)}\cdot\nabla\bar{F}_i - \dot{V}_\parallel^{(1)}\frac{\partial\bar{F}_i}{\partial v_\parallel} + C[\delta F_i], \quad (40a)$$

where we have defined

$$\dot{\mathbf{X}} = v_\parallel\frac{\mathbf{B}^*}{B_\parallel^*} + \frac{m_i c}{eB_\parallel^*}\hat{b}\times\left[\frac{e}{m_i}\nabla(\langle\delta\phi\rangle_\alpha + \bar{\phi}) + \mu\nabla B\right], \quad (40b)$$

$$\dot{V}_\parallel = -\frac{\mathbf{B}^*}{B_\parallel^*}\cdot\left[\frac{e}{m_i}\nabla(\langle\delta\phi\rangle_\alpha + \bar{\phi}) + \mu\nabla B\right], \quad (40c)$$

with

$$\mathbf{B}^* \equiv \mathbf{B} + v_\parallel\frac{m_i c}{e}\nabla\times\hat{b}. \quad (40d)$$

Here $B_\parallel^* \equiv \hat{b}\cdot\mathbf{B}^*$ and the gyroangle average is defined by $\langle\cdots\rangle_\alpha \equiv (2\pi)^{-1}\oint d\alpha(\cdots)$. Operating on Eq. (40a) with $2\pi\int d\mu dv_\parallel B_\parallel^*$ yields

$$\begin{aligned}
&\frac{\partial\delta N_i}{\partial t} + \frac{B}{\omega_{ci}}(\nabla\times\hat{b})\cdot\nabla_\perp\left(2\pi\int d\mu dv_\parallel B v_\parallel^2\frac{\delta F_i}{B}\right) \\
&+ \frac{B^2}{\omega_{ci}}(\hat{b}\times\nabla_\perp\ln B)\cdot\nabla_\perp\left(2\pi\int d\mu dv_\parallel B\mu B\frac{\delta F_i}{B^2}\right) \\
&+ 2\pi\frac{m_i c}{e}\int d\mu dv_\parallel \mu\delta F_i(\nabla\times\hat{b})\cdot\nabla B \\
&= c(\hat{b}\times\nabla_\perp\delta\phi)\cdot\nabla_\perp\left(2\pi\int d\mu dv_\parallel B\frac{\bar{F}_i}{B}\right) \\
&+ c(\hat{b}\times\nabla_\perp\bar{\phi})\cdot\nabla_\perp\left(2\pi\int d\mu dv_\parallel B\frac{\delta F_i}{B}\right) \\
&- c(\nabla\times\hat{b})\cdot\nabla\delta\phi\left(2\pi\int d\mu dv_\parallel B\frac{\bar{F}_i}{B}\right) + D\nabla_\perp^2\delta N_i,
\end{aligned} \quad (41)$$

where

$$N_i \equiv 2\pi\int d\mu dv_\parallel B_\parallel^* F_i \approx 2\pi\int d\mu dv_\parallel B F_i$$

is the gyrocenter density, for simplicity we have taken $\hat{b}\cdot\nabla \to 0$ and assumed the long wavelength limit such that $J_0(\lambda) \approx 1$. Also, for convenience we have assumed that the collisional transport can be modeled as diffusive, i.e., we have taken $\int d^3\bar{v}C[\delta F_i] = D\nabla_\perp^2\delta N_i$, where D is a collisional diffusion coefficient which is taken to be a constant. For $\beta \ll 1$, the fourth term on the left hand side of Eq. (41) can be written as

$$2\pi\frac{m_i c}{e}\int d\mu dv_\parallel \delta F_i(\nabla\times\hat{b})\cdot\nabla B \approx 2\pi\frac{m_i c}{e}\int d\mu dv_\parallel\delta F_i(\hat{b}\times\nabla_\perp\ln B)\cdot\nabla B = 0, \quad (42)$$

where we have noted that $\nabla\times\hat{b} \approx \hat{b}\times\nabla_\perp\ln B$ for low-β plasmas. Neglecting this contribution, Eq. (41) can be written as

$$\begin{aligned}
&\frac{\partial\delta N_i}{\partial t} + \frac{\nabla\times\hat{b}}{m_i\omega_{ci}}\cdot\nabla\delta P_{\parallel i} + \frac{\hat{b}\times\nabla_\perp\ln B}{m_i\omega_{ci}}\cdot\nabla_\perp\delta P_{\perp i} \\
&= c(\hat{b}\times\nabla_\perp\delta\phi)\cdot\nabla_\perp\left(\frac{\bar{N}_i}{B}\right) + c(\hat{b}\times\nabla_\perp\bar{\phi})\cdot\nabla_\perp\left(\frac{\delta N_i}{B}\right) \\
&- c(\nabla\times\hat{b})\cdot\nabla\delta\phi\left(\frac{\bar{N}_i}{B}\right) + D\nabla_\perp^2\delta N_i,
\end{aligned} \quad (43)$$

where

$$P_{\parallel i} \equiv 2\pi m_i\int d\mu dv_\parallel B_\parallel^*(v_\parallel - U_\parallel)^2 F_i \approx 2\pi m_i\int d\mu dv_\parallel B(v_\parallel - U_\parallel)^2 F_i,$$

$$P_{\perp i} \equiv 2\pi m_i\int d\mu dv_\parallel B_\parallel^*\mu B F_i \approx 2\pi m_i\int d\mu dv_\parallel B\mu B F_i,$$

$$N_i U_\parallel \equiv 2\pi\int d\mu dv_\parallel B_\parallel^* v_\parallel F_i \approx 2\pi\int d\mu dv_\parallel B v_\parallel F_i,$$

and we have made the approximation

$$\begin{aligned}
2\pi&\int d\mu dv_\parallel B(v_\parallel - U_\parallel)^2\delta F_i \\
&= 2\pi\int d\mu dv_\parallel B\,\delta F_i[(v_\parallel - U_\parallel)^2 + 2U_\parallel v_\parallel - U_\parallel^2] \\
&= \frac{\delta P_{\parallel i}}{m_i} + 2U_\parallel\delta(N_i U_\parallel) - U_\parallel^2\delta N_i \approx \frac{\delta P_{\parallel i}}{m_i}
\end{aligned} \quad (44)$$

valid for low Mach number flows. Equation (43) can be rewritten in the suggestive form given by

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\delta N_i}{B^2}\right) &= \delta\mathbf{u}_\perp^{EB}\cdot\nabla_\perp\left(\frac{\bar{N}_i}{B^2}\right) + \overline{\mathbf{u}_\perp^{EB}}\cdot\nabla_\perp\left(\frac{\delta N_i}{B^2}\right) \\
&- \left(\frac{\hat{b}\times\nabla_\perp\ln B}{m_i\omega_{ci}B^2}\right)\cdot\nabla_\perp(\delta P_{\parallel i} + \delta P_{\perp i}) \\
&+ D\nabla_\perp^2\left(\frac{\delta N_i}{B^2}\right),
\end{aligned} \quad (45)$$

where we have temporarily assumed the grad-B drift to be equivalent to the curvature drift. Equation (45) bears a close resemblance to Eq. (3) with the magnetically weighted gyrocenter density acting as the effective potential vorticity of the system, and the driving terms replaced by grad-B induced coupling to pressure fluctuations. In physical coordinates the ion gyrocenter density can be approximated as

$$\frac{\overline{N_i}}{B^2} \approx \frac{n_0}{B^2} + \frac{1}{B^2}\nabla_\perp \cdot \left(m_i n_0 \frac{c^2}{eB^2}\overline{\mathbf{E}}_\perp \right) - \frac{1}{2}\frac{1}{B^2}\nabla_\perp \cdot (n_0 \rho_i^2 \nabla_\perp \ln \overline{P_i}) + \cdots .$$

Note that the first two terms in the above expression correspond to the (magnetically weighted) density minus the vorticity, reminiscent of expressions for potential vorticity within reduced fluid models usch as the Hasegawa-Wakatani equation. The remaining terms result from $T_i/T_e \neq 0$, and thus this quantity can thus be recognized as providing a natural generization of potential vorticity to the gyrokinetic formulation. In the following, we will exploit this analogy and refer to the ion cyrocenter density as an effective potential vorticity. It is hoped that this analogy will help further physical connections between highly idealized fluid models and the somewhat technical gyrokinetic equations.

Decomposing the pressure fluctuations as $\delta P_i = \delta(N_i T_i) \approx \delta N_i T_i + \bar{N}_i \delta T_i$, and Fourier transforming, allows Eq. (43) to be written as

$$\delta N_{i,k} = (\Omega_k + i\nu_k)^{-1} \Bigg\{ c\left[(\hat{b}\times\mathbf{k}_\perp)\cdot\nabla\left(\frac{\bar{N}_i}{B}\right) + (\nabla\times\hat{b})\cdot\mathbf{k}_\perp\left(\frac{\bar{N}_i}{B}\right)\right]\delta\phi_k + \frac{c}{eB}\bar{N}_i[(\nabla\times\hat{b})\cdot\mathbf{k}\,\delta T_{\parallel i,k} + (\hat{b}\times\nabla_\perp \ln B)\cdot\mathbf{k}\,\delta T_{\perp i,k}] \Bigg\}, \tag{46}$$

where we have defined $\nu_k \equiv D k_\perp^2$, $\Omega_k \equiv \omega_k - \omega_{d\kappa} - \omega_{d\nabla B}$,

$$\omega_{d\kappa} \equiv \frac{T_{\parallel i}}{m_i \omega_{ci}}(\nabla\times\hat{b})\cdot\mathbf{k}, \tag{47a}$$

$$\omega_{d\nabla B} \equiv \frac{T_{\perp i}}{m_i \omega_{ci}}(\hat{b}\times\nabla_\perp \ln B)\cdot\mathbf{k}, \tag{47b}$$

and the Doppler shift induced by the mean $E\times B$ flow has been absorbed into ω_k. Substituting Eq. (46) into Eq. (39) yields

$$n_i m_i \mu_{ii}^{(\text{neo})}(u_\theta - u_\theta^{\text{neo}}) = m_i \omega_{ci}(\Gamma_N - \Gamma_n), \tag{48a}$$

where Γ_n is the particle flux, and the gyrocenter flux has been approximated by

$$\Gamma_N = -c_s^2\rho_s^2 B \,\mathrm{Re}\sum_k \frac{i(\hat{b}\times\mathbf{k}_\perp)_r}{\Omega_k + i\nu_k}\left|\frac{e\delta\phi_k}{T_e}\right|^2 \times\left[(\hat{b}\times\mathbf{k}_\perp)\cdot\nabla\left(\frac{\bar{N}_i}{B}\right) + (\nabla\times\hat{b})\cdot\mathbf{k}_\perp\left(\frac{\bar{N}_i}{B}\right)\right] - \bar{N}_i c_s^2\rho_s^2\,\mathrm{Re}\sum_k \frac{i(\hat{b}\times\mathbf{k}_\perp)_r}{\Omega_k + i\nu_k} \times\left[(\nabla\times\hat{b})\cdot\mathbf{k}\frac{e\delta\phi_{-k}}{T_e}\frac{\delta T_{\parallel i,k}}{T_e} + (\hat{b}\times\nabla_\perp \ln B)\cdot\mathbf{k}_\perp\frac{e\delta\phi_{-k}}{T_e}\frac{\delta T_{\perp i,k}}{T_e}\right]. \tag{48b}$$

It is useful to discuss Eq. (48b) in a number of limits. First considering the simple limit of B=const and $T_i/T_e \to 0$. In this limit, $\overline{N_i} = \bar{n}_i - \nabla_\perp \cdot [m_i \bar{n}_i (c^2/eB^2)\nabla_\perp \bar{\phi}]$, such that Eq. (48b) can be written as

$$\Gamma_N = -D_{\text{PV}}\frac{\partial}{\partial r}\left[\bar{n}_i - \nabla_\perp\cdot\left(m_i \bar{n}_i \frac{c^2}{eB^2}\nabla_\perp \bar{\phi}\right)\right], \tag{49}$$

where

$$D_{\text{PV}} \equiv c_s^2 \rho_s^2 \,\mathrm{Re}\sum_k \frac{i}{\Omega_k + i\nu_k}(\hat{b}\times\mathbf{k}_\perp)_r^2\left|\frac{e\delta\phi_k}{T_e}\right|^2 .$$

Thus, in this simple limit the gyrocenter flux can be seen to correspond to the diffusive flux of potential vorticity, which in this idealized limit is given by $\bar{n}_i - \nabla_\perp \cdot [m_i \bar{n}_i c^2/(eB^2)\nabla_\perp \bar{\phi}]$.

If we now consider a magnetic field of the form $B = B(r)$, but assume $T_i/T_e \to 0$, Eq. (48b) can be reduced to

$$\Gamma_N = -c_s^2\rho_s^2 B \,\mathrm{Re}\sum_k \frac{i(\hat{b}\times\mathbf{k}_\perp)_r}{\Omega_k + i\nu_k}\left|\frac{e\delta\phi_k}{T_e}\right|^2 \times\left[(\hat{b}\times\mathbf{k}_\perp)\cdot\nabla\left(\frac{\bar{N}_i}{B}\right) + (\nabla\times\hat{b})\cdot\mathbf{k}_\perp\left(\frac{\bar{N}_i}{B}\right)\right]. \tag{50}$$

Hence, inhomogeneities in the equilibrium magnetic field can be seen to drive nondiffusive contributions to the PV flux. For the specific case of a straight magnetic field described by $\mathbf{B} = B(r)\hat{z}$, such that the second term in brackets in Eq. (50) vanishes, Eq. (50) can be written as

$$\Gamma_N = -D_{\text{PV}} B \frac{\partial}{\partial r}\left(\frac{\bar{N}_i}{B}\right). \tag{51}$$

Thus, the diffused quantity for this system can be identified as the potential vorticity weighted by the magnetic field. For the more general case of a curved magnetic field, but with $\beta \ll 1$ such that the curvature and grad-B drifts are approximately equal, Eq. (48b) can be written as

$$\Gamma_N = -D_{\text{PV}} B^2 \frac{\partial}{\partial r}\left(\frac{\bar{N}_i}{B^2}\right). \tag{52}$$

For this more general case, deviations from neoclassical rates of rotation are now linked to the mean PV weighted by B^2. Hence, in analogy with particle and toroidal momentum transport, a TEP contribution is also present in the PV flux. As noted by Refs. 33 and 41–46 TEP contributions arise as a consequence of the compressibility of the $E\times B$ drift motion and are hence anticipated to be a robust component of the particle, parallel momentum, or in this case, PV flux. The specific power of the magnetic field present within the TEP pinch is not universal (as noted above), but is often determined by the relation $\nabla_\perp\cdot(B^\alpha\delta\mathbf{u}^{EB})=0$. For a straight magnetic field it is easy to see that α should be chosen to be unity, whereas for low-β tokamak plasmas $\alpha=2$ is often a good approximation.

Finally, in the limit of finite T_i/T_e, but assuming isotropic temperature fluctuations for simplicity, Eq. (48a) can be written as (also assuming a low-β curved magnetic field topology)

$$u_\theta=u_\theta^{\text{neo}}-\frac{\omega_{ci}}{\mu_{ii}^{(\text{neo})}}\left[D_{\text{PV}}\frac{B^2}{n_0}\frac{\partial}{\partial r}\left(\frac{\bar{N}_i}{B^2}\right)+V_{\text{PV}}^{\text{th}}\frac{\bar{N}_i}{n_0}+\frac{\Gamma_n}{n_0}\right], \tag{53a}$$

$$V_{\text{PV}}^{\text{th}}\equiv-2\ \text{Re}\sum_k\frac{i\omega_{d\nabla B}}{\Omega_k+i\nu_k}\delta u_{r,-k}^{(EB)}\frac{\delta T_{i,k}}{T_i}, \tag{53b}$$

where the second term in brackets will be referred to as a thermoelectric contribution. Written in this form it is clear that deviations from neoclassical rotation can be linked to diffusion of the magnetically weighted PV, thermoelectric pinch of PV, or particle fluxes. In order to investigate the contribution from the thermoelectric pinch it is useful to consider a linear expression for isotropic temperature perturbations given by

$$\frac{\delta T_{i,k}}{T_i}=[\omega_k-(14/3)\omega_{d\nabla B}+i\nu_k]^{-1}\left\{\left[-\omega_{Ti}^*+\frac{4}{3}\omega_{d\nabla B}\right]\frac{e\delta\phi_k}{T_i}+\frac{4}{3}\omega_{d\nabla B}\frac{\delta N_{i,k}}{n_0}\right\}, \tag{54}$$

where

$$\omega_{Ti}^*\equiv v_{thi}\rho_i(\hat{b}\times\nabla_\perp\ \ln T_i)\cdot\mathbf{k}, \tag{55a}$$

$$\omega_{Ni}^*\equiv v_{thi}\rho_i(\hat{b}\times\nabla_\perp\ \ln N_i)\cdot\mathbf{k}. \tag{55b}$$

Substituting Eq. (54) into Eq. (53b) yields

$$V_{\text{PV}}^{\text{th}}=-2\ \text{Re}\sum_k\frac{i\omega_{d\nabla B}^2}{\omega_k-2\omega_{d\nabla B}+i\nu_k}\cdot\frac{\delta u_{r,-k}^{(EB)}}{\omega_k-(14/3)\omega_{d\nabla B}+i\nu_k}\times\left\{\frac{4}{3}\frac{\delta N_{i,k}}{n_0}+\left(\frac{4}{3}-\frac{\omega_{Ti}^*}{\omega_{d\nabla B}}\right)\frac{e\delta\phi_k}{T_i}\right\}. \tag{56}$$

The gyrocenter density can be written in physical variables for the limit of adiabatic electrons as

$$\delta N_{i,k}\approx\delta n_k+n_0\rho_s^2k_\perp^2\frac{e\delta\phi_k}{T_e}=n_0(1+\rho_s^2k_\perp^2)\frac{e\delta\phi_k}{T_e}, \tag{57}$$

such that Eq. (56) can be written as

$$V_{\text{PV}}^{\text{th}}=-2\ \text{Re}\sum_k\frac{i\omega_{d\nabla B}^2}{\omega_k-2\omega_{d\nabla B}+i\nu_k}\cdot\frac{\delta u_{r,-k}^{(EB)}}{\omega_k-(14/3)\omega_{d\nabla B}+i\nu_k}\frac{e\delta\phi_k}{T_e}\times\left[\frac{4}{3}(1+\tau+\rho_s^2k_\perp^2)-\tau\frac{\omega_{Ti}^*}{\omega_{d\nabla B}}\right]. \tag{58}$$

Equation (53a), along with Eq. (58), provides an explicit expression for the quasilinear estimate of the poloidal plasma rotation. Our primary motivation up to this point has been to emphasize the physical origin of the various components of the PV flux. In order to derive a more compact expression for the poloidal rotation it is useful to utilize the linear dispersion relation for the above system, i.e.,

$$D_{k,\omega}=1+k_\perp^2\rho_s^2+\tau\left(\frac{\omega_{Ni}^*-2\omega_{d\nabla B}}{\omega-2\omega_{d\nabla B}+i\nu_k}\right)-\frac{2\tau\omega_{d\nabla B}^2}{(\omega-2\omega_{d\nabla B}+i\nu_k)\left(\omega-\frac{14}{3}\omega_{d\nabla B}+i\nu_k\right)}\times\left\{-\frac{\omega_{Ti}^*}{\omega_{d\nabla B}}+\tfrac{4}{3}[1+\tau^{-1}(1+k_\perp^2\rho_s^2)]\right\}. \tag{59}$$

From inspection of Eqs. (53a), (58), and (59), also noting the definitions given by Eqs. (47b) and (55), it is apparent that the poloidal flow can be written in the compact form given by (assuming adiabatic electrons)

$$u_\theta=u_\theta^{(\text{neo})}+\frac{c_s^2}{\mu_{ii}^{(\text{neo})}}\sum_k k_y\ \text{Im}\ D_{k,\omega_k}\left|\frac{e\delta\phi_k}{T_e}\right|^2. \tag{60}$$

Thus, at stationarity the turbulent stress exerted on the poloidal flow is linked to the dissipation/growth profile. This result can be seen to be a direct analog of Eq. (6) above.

B. Kinetic limit

Within this subsection we focus on the case of turbulence near marginality for low collisionality regimes, i.e., $\nu_k\to 0$. Within this regime the PV flux will take on a highly resonant character in contrast to the fluid limit considered above. While in Sec. IV A it was convenient to work in the idealized limit of $k_\parallel=0$, $T_i/T_e\ll 1$, and adiabatic electrons, here we will relax these constraints such that a more general result may be derived. Rather than working with the total gyrocenter phase space perturbation δF_i, it will be convenient to separate δF_i into adiabatic and nonadiabatic pieces, i.e.,

$$\delta F_{i,k}=-J_0\frac{e\delta\phi_k}{T_i}\overline{F}_i+\delta G_{i,k}. \tag{61}$$

With this definition Eq. (40) can be written as

$$(\omega_k-v_\parallel k_\parallel-\omega_D)\delta G_{i,k}=(\omega_k-\omega^*)\overline{F}_iJ_0\frac{e\delta\phi_k}{T_i}, \tag{62a}$$

where

$$\omega_D \equiv \frac{v_\parallel^2}{\omega_{ci}}(\nabla \times \hat{b}) \cdot \mathbf{k} + \frac{\mu B}{\omega_{ci}}(\hat{b} \times \nabla \ln B) \cdot \mathbf{k}_\perp, \tag{62b}$$

$$\omega^* \equiv - v_{thi}\rho_i\left[1 + \left(\frac{v_\parallel^2/2 + \mu B}{v_{thi}^2} - \frac{3}{2}\right)\eta_i\right](\hat{b} \times \mathbf{k}_\perp) \cdot \nabla \ln \bar{N}_i. \tag{62c}$$

Utilizing Eq. (62a), the gyrocenter flux may be estimated as

$$\Gamma_N - \Gamma_n = v_{thi}\rho_i \text{ Re} \sum_k ik_y \int d^3\bar{v} J_0^2 \overline{F}_i \left(\frac{\omega_k - \omega^*}{\omega_k - v_\parallel k_\parallel - \omega_D}\right) \times \left|\frac{e\delta\phi_k}{T_i}\right|^2 - v_{thi}\rho_i \text{ Re} \sum_k ik_y \frac{e\delta\phi_{-k}}{T_i}\delta n_k^{NA}. \tag{63}$$

In the resonant limit considered here, Eq. (63) can be written as

$$\Gamma_N - \Gamma_n = \pi v_{thi}\rho_i \sum_k k_y \left|\frac{e\delta\phi_k}{T_i}\right|^2 \int d^3\bar{v} J_0^2 \overline{F}_i \delta(\omega_k - v_\parallel k_\parallel - \omega_D)(\omega - \omega^*) + v_{thi}\rho_i \frac{T_i}{e}\sum_k k_y \left|\frac{e\delta\phi_k}{T_i}\right|^2 \text{Im}\left(\frac{\delta n_k^{NA}}{\delta\phi_k}\right). \tag{64}$$

While this flux may be evaluated directly via the use of approximations such as the ∇B model,[47,48] it is useful to utilize the plasma dispersion relation directly. For the limit considered here ($k_\perp^2\rho_i^2 < 1$), the plasma dispersion relation may be written as

$$D_{k,\omega} = 1 + k_\perp^2\rho_s^2 + \tau\Gamma_0 - \frac{\tau}{\overline{N}_i}\int d^3\bar{v} J_0^2 \overline{F}_i \left(\frac{\omega - \omega^*}{\omega - v_\parallel k_\parallel - \omega_D}\right) + \frac{T_e}{e\overline{N}_i}\frac{\delta n_{k,\omega}^{NA}}{\delta\phi_{k,\omega}}, \tag{65}$$

where $\Gamma_0 \equiv I_0(k_\perp^2\rho_i^2)e^{-k_\perp^2\rho_i^2}$ and I_0 is a modified Bessel function. Note that while both toroidal and acoustic ion temperature gradient instabilities are contained within Eq. (65),[49,50] for simplicity we have assumed the mean parallel velocity to be zero such that the parallel velocity shear instability[51,52] is not present. The imaginary component can be written in the form

$$\text{Im } D_{k,\omega} = \pi\frac{\tau}{\overline{N}_i}\int d^3\bar{v} J_0^2 \overline{F}_i \delta(\omega - v_\parallel k_\parallel - \omega_D)(\omega - \omega^*) + \frac{T_e}{e\overline{N}_i}\text{Im}\left(\frac{\delta n_{k,\omega}^{NA}}{\delta\phi_{k,\omega}}\right). \tag{66}$$

Utilizing Eq. (66), Eq. (64) may be written in the compact form given by

$$\begin{aligned}\Gamma_N - \Gamma_n &= \tau^{-1} v_{thi}\rho_i \overline{N}_i \sum_k k_y \text{ Im } D_{k,\omega_k}\left|\frac{e\delta\phi_k}{T_i}\right|^2 \\ &= -\tau^{-1} v_{thi}\rho_i \overline{N}_i \sum_k \gamma_k k_y \left.\frac{\partial D_{k,\omega}}{\partial\omega}\right|_{\omega_k}\left|\frac{e\delta\phi_k}{T_i}\right|^2 \\ &= -\frac{2}{m_i\omega_{ci}}\sum_k \gamma_k k_y \frac{E_k}{\omega_k},\end{aligned} \tag{67}$$

where we have used

$$\gamma_k = \left.\frac{-\text{Im } D_{k,\omega}}{\partial D_{k,\omega}/\partial\omega}\right|_{\omega_k},$$

and noted the definition

$$E_k = \frac{1}{2}\overline{N}_i T_e \omega_k \left.\frac{\partial D_{k,\omega}}{\partial\omega}\right|_{\omega_k}\left|\frac{e\delta\phi_k}{T_e}\right|^2.$$

An expression for the poloidal mean flow may be derived at stationarity by substituting Eq. (67) into Eq. (48a), yielding

$$u_\theta = u_\theta^{\text{neo}} - 2\frac{1}{n_i m_i}\frac{1}{\mu_{ii}^{(\text{neo})}}\sum_k \gamma_k k_y \frac{E_k}{\omega_k}. \tag{68}$$

Equation (68) can be seen to have a similar structure to Eq. (6) above. Here, the generation of poloidal flow is again linked to both the sign of the underlying wave momentum as well as the local stability profile. Note that while we have neglected both collisions and nonlinearity within the wave dynamics, the overlap of wave-particle resonances provides the requisite means of irreversible momentum transport.

V. CONCLUSION AND DISCUSSION

In this analysis, turbulence driven deviations from poloidal rotation have been discussed within the context of a population of small amplitude waves in the limit of an axisymmetric large aspect ratio plasma. The primary results of this analysis are as follows:

(a) A kinetic analog of a Taylor identity has been derived in the small inverse aspect ratio limit for the long wavelength limit (Sec. III), and in slab geometry for general $k_\perp^2\rho_i^2$ (Appendix B).
(b) The flux of potential vorticity has been evaluated via quasilinear theory, revealing diffusive, TEP, and thermoelectric contributions.
(c) At stationarity, deviations from neoclassical predictions of poloidal rotation have been shown to be linked to regions of emission and absorption of wave momentum, and hence are closely linked to the stability profile of the underlying fluctuations.

It is useful at this point to discuss the relation of items (a)–(c) to existing results within the fluid and plasma literature. Considering (a) first, it is useful to recall the explicit form of the fluid Taylor identity given by Eq. (2) above (in dimensional units),

$$\overline{(1-\rho_s^2\nabla_\perp^2)\frac{e\delta\phi}{T_e}\frac{c}{B}(\hat{b}\times\nabla_\perp\delta\phi)_x}$$
$$=-\frac{1}{\omega_{ci}}\frac{c^2}{B^2}\overline{\nabla_\perp^2\delta\phi(\hat{b}\times\nabla_\perp\delta\phi)_x}.$$

After an integration by parts this expression can be written as

$$\frac{c^2}{B^2}\overline{\nabla_\perp^2\delta\phi(\hat{b}\times\nabla_\perp\delta\phi)_x}=\frac{\partial}{\partial x}\overline{\delta u_x^{(EB)}\delta u_y^{(EB)}}, \tag{69}$$

where we have assumed a simplified slab geometry and taken $B=\text{const}$. We now consider the slab analog of Eq. (36) above, which can be written as (see Appendix B)

$$\overline{\sum_s q_s\int d^3\bar{v}\,\delta F_s(\hat{b}\times\nabla_\perp J_0\delta\phi)_x}=-\frac{\partial}{\partial x}\left[m_i\frac{c}{B}\overline{\delta(n_iu_y)\delta E_y}\right], \tag{70a}$$

where in the long wavelength limit the gyrocenter density is constrained by (for $B=\text{const}$)

$$m_i\frac{c^2}{B^2}\nabla_\perp\cdot(n_0\nabla_\perp\delta\phi)=-\sum_s q_s\int d^3\bar{v}J_0\delta F_s. \tag{70b}$$

Comparing the form of Eq. (69) with Eq. (70) it is clear that these relations have a similar structure. Namely, the LHS of Eq. (70a) describes the radial mixing of gyrocenter density, which is in turn constrained by the plasma polarization density (vorticity) through Eq. (70b). While for simple limits such as $T_i/T_e\to 0$, such that $J_0\to 1$, Eqs. (70a) and (70b) reduce to Eq. (69), the turbulent stress on the RHS of Eq. (70a) includes the total perpendicular velocity (i.e., diamagnetic and finite Larmor radius corrections in addition to the $E\times B$ velocity). Hence, Eq. (70a) can be seen as a natural extension of a Taylor identity to a kinetic description of a strongly magnetized plasma.

Similarly, it is instructive to compare the results derived in Ref. 11 (referred to hereafter as DK91) with Eq. (70a). DK91 describes the generation of mean poloidal flow via a quasilinear calculation of the radial current. More explicitly, the radial current can be linked to the mean poloidal flow via the momentum equation given by their Eq. (15) (rewritten here at stationarity using the notation of this analysis),

$$\mu_{ii}^{(\text{neo})}n_im_i(\overline{u_\theta}-u_\theta^{(\text{neo})})=-\frac{J_xB_\varphi}{c}, \tag{71}$$

where the radial current is defined as

$$J_x\equiv e\overline{(\delta n_i-\delta n_e)\delta u_x^{(EB)}}.$$

After exploiting the Poynting theorem for the poloidal wave momentum, this expression is linked to the divergence of the wave stress, namely,

$$\frac{J_xB_\varphi}{c}=\frac{\partial}{\partial x}\sum_k v_{gr}k_y\frac{E_k}{\omega_k}. \tag{72}$$

If we now consider Eq. (70a), this expression can be rewritten in a more suggestive form as

$$\overline{\sum_s q_s\int d^3\bar{v}\,\delta F_s(\hat{b}\times\nabla_\perp J_0\delta\phi)_x}$$
$$=\frac{eB}{c}\overline{(\delta\hat{N}_i-\delta n_e)\delta u_x^{(EB)}}$$
$$=-\frac{\partial}{\partial x}\left[m_i\frac{c}{B}\overline{\delta(n_iu_y)\delta E_y}\right], \tag{73}$$

where $\delta\hat{N}_i\equiv\int d^3\bar{v}\,\delta F_iJ_0$ and we note that $\delta\hat{N}_i$ is an operator. If we define a mean radial current as

$$J_x^{(\text{pol})}\equiv-e\overline{(\delta\hat{N}_i-\delta n_e)\delta u_x^{(EB)}},$$

this allows Eq. (73) to be rewritten in the compact form,

$$\frac{J_x^{(\text{pol})}B}{c}=\frac{\partial}{\partial x}\left[m_i\frac{c}{B}\overline{\delta(n_iu_y)\delta E_y}\right]. \tag{74}$$

Here the minus sign in the definition of $J_x^{(\text{pol})}$ is due to this current corresponding to a flux of polarization charge. Using this relation, the poloidal momentum equation can be written in the form

$$\mu_{ii}^{(\text{neo})}n_im_i(\overline{u_\theta}-u_\theta^{(\text{neo})})=-\frac{J_x^{(\text{pol})}B}{c}, \tag{75}$$

consistent with Eq. (71). Note that while the relation between the mean radial current and wave stress given in DK91 [our Eq. (72)] was derived in a highly simplified limit (i.e., assuming small amplitude fluctuations, neglecting diamagnetic terms as well as finite Larmor radius corrections), Eq. (74) applies to a much more general set of plasma phenomena (see Appendix B). Hence, while DK91 utilized a highly idealized model of drift wave turbulence, it is clear that their analysis is likely relevant to a wider range of parameter regimes than a strict interpretation of the original derivation would imply.

It is useful at this point to compare the previous results derived within the context of parallel momentum transport to the results discussed here. Reference 36 utilized an analysis similar to that discussed in Sec. III in order to derive the expression

$$\overline{\sum_s q_s\int d^3\bar{v}\,\delta F_s\hat{b}\cdot\nabla J_0\delta\phi}$$
$$\approx-\frac{\partial}{\partial x}\overline{\left(n_0m_i\frac{c}{B}\delta E_x-\sum_s\text{sgn}(q_s)m_s\int d^3\bar{v}\frac{\mu B}{\omega_{cs}}\frac{\partial\delta F_s}{\partial x}\right)\frac{c}{B}\delta E_\parallel}, \tag{76}$$

where for simplicity we again consider slab geometry. We note that this expression was derived in the long wavelength limit, hence we can again utilize the long wavelength expression for perpendicular momentum given by Eq. (35). Equation (76) can then be written in the compact form given by

$$\overline{\sum_s q_s\int d^3\bar{v}\,\delta F_s\hat{b}\cdot\nabla J_0\delta\phi}\approx\frac{\partial}{\partial x}\left[m_i\frac{c}{B}\overline{\delta(n_iu_y)\delta E_\parallel}\right]. \tag{77}$$

Equation (77) has a closely analogous form to Eq. (70a), and it is thus convenient to refer to this relation as a parallel analog of a Taylor identity. The presence of this additional constraint can be seen to follow as a result of the presence of *two* directions of translational symmetry (i.e., the parallel and zonal directions) being present within the simplified geometry utilized above. This is in contrast to the single symmetry direction present within many familiar examples from geophysical fluid dynamics. While no small amplitude assumption was utilized in the derivation of Eqs. (77) and (70a), for the limit of small amplitude waves the right hand sides of Eqs. (77) and (70a) correspond to parallel and perpendicular wave stresses (see the discussion at the beginning of Sec. III). Hence, the above relations can be seen to link the perpendicular and parallel wave stresses to the mixing of a 4D gyrocenter fluid in real and velocity space, respectively.

With regard to (b), the presence of a Taylor identity for a strongly magnetized plasma allows the perpendicular stress divergence to be explicitly linked to the flux of the gyrocenter density. Heuristically, as noted above, the gyrocenter density can be seen to correspond to a generalization of the concept of potential vorticity to a variety of models of plasma turbulence. Thus, the turbulent stress exerted on the mean flow can be seen to be closely linked to the transport of potential vorticity. While the link between the transport of potential vorticity and perpendicular stress divergence has long been appreciated within the geophysical literature (see Ref. 53 and references therein), the extension of this concept to a strongly magnetized plasma is, however, relatively unexplored (see Ref. 20). A surprising result of the quasilinear analysis employed above is that the presence of an inhomogeneous magnetic field provides a possible means of transporting the effective potential vorticity of the system *up* the mean gradient via the thermoelectric pinch of potential vorticity. In order to better understand the origin of this potential up-gradient transport, it is useful to compare a simplified version of the potential vorticity equation in a strongly magnetized plasma with its geophysical counterpart. For a geophysical fluid, potential vorticity can be conveniently defined by the relation $\Pi \equiv (\omega/\rho)\cdot\nabla\lambda$, where ω is the vorticity which generally includes both relative and planetary components, ρ is the density, and λ is a scalar fluid quantity which satisfies $d\lambda/dt=0$. With these definitions it is straightforward to derive the relation[54]

$$\frac{d}{dt}\left(\frac{\omega}{\rho}\cdot\nabla\lambda\right) = \nabla\lambda\cdot\left(\frac{\nabla\rho\times\nabla P}{\rho^3}\right), \tag{78}$$

where P is the fluid pressure and we have neglected all dissipation. Thus, the potential vorticity can be seen to be conserved for either a baroclinic flow, i.e., $\nabla\rho\times\nabla P=0$, or for $\lambda=\lambda(\rho,P)$ such that $\nabla\lambda$ is perpendicular to the baroclinic vector $\nabla\rho\times\nabla P$. Similarly, following an analogous set of procedures leading up to the derivation of Eq. (45), but without linearization, the gyrokinetic analog of Eq. (78) can be written as

$$\frac{d}{dt}\left(\frac{N_i}{B^2}\right) = \hat{b}\cdot\left(\frac{\nabla P_i\times\nabla\ln B}{m_i\omega_{ci}B^2}\right), \tag{79}$$

where we have defined $d/dt=\partial/\partial t+\mathbf{u}_\perp^{EB}\cdot\nabla$ and assumed $\hat{b}\cdot\nabla\to 0$, $\nabla\times\hat{b}\approx\hat{b}\times\nabla\ln B$, $J_0\to 1$, $C[F]\to 0$ as well as an isotropic temperature for simplicity. Equation (79) can be seen to have a closely analogous form as Eq. (78) but with the baroclinic vector replaced by the grad-B drift. As shown above, the thermoelectric terms in the potential vorticity flux result from the RHS of Eq. (78) [see Eqs. (43)–(46) above]. Thus, the up-gradient flux of potential vorticity can be seen to result from the violation of potential vorticity conservation.

Considering (c), the enforcement of self-consistency via the quasineutrality relation provides an explicit link between the stability profile of the underlying fluctuations and deviations from neoclassical predictions of poloidal rotation. The underlying physics of this was discussed within the context of a small amplitude nonacceleration theorem within Sec. I. However, here it is useful to provide an additional perspective on this result. As discussed in Sec. I, the sensitivity of the mean flow to the forcing and dissipation profile can be understood to follow as a result of the emission and absorption of wave momentum in regions above and below marginality (respectively). In this context it is natural to consider whether an analogous result can be obtained via the consideration of a Poynting theorem for wave momentum,[11,55,56] i.e.,

$$\frac{\partial P_y^w}{\partial t} + \frac{\partial}{\partial x}\sum_k v_{gr}k_y\frac{E_k}{\omega_k} = 2\sum_k \gamma_k k_y\frac{E_k}{\omega_k}, \tag{80}$$

where $P_y^w \equiv \Sigma_k k_y E_k/\omega_k$. In the limit of stationary waves, we have

$$\frac{\partial}{\partial x}\sum_k v_{gr}k_y\frac{E_k}{\omega_k} = 2\sum_k \gamma_k k_y\frac{E_k}{\omega_k}, \tag{81}$$

hence, consistent with elementary considerations, one again sees a clear correspondence between the divergence of the wave stress and the emission and absorption of wave momentum. This close association of turbulent stresses with local stability properties suggests that the turbulent generation of poloidal flows can be seen to be closely correlated with the evolution of the mean density and temperature profiles.

Of primary interest in the study of poloidal rotation is understanding the impact of rotational shear on the formation of internal transport barriers.[57,58] In this regard, a description of possible stable flow profiles should provide insight into what role poloidal rotation can play in both triggering as well as sustaining transport barriers. Future work will be oriented toward the development of a self-consistent transport model to better understand the types of flow structures which can be described within the framework of this simple description of turbulently driven poloidal flows. In particular, special emphasis will be placed on the description of poloidal spin-up near transport barriers.

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APPENDIX A: DERIVATION OF EQUATION FOR STRESS TENSOR

In this section we will be interested in deriving transparent expressions for the components of the stress tensor $\mathbf{\Pi}$. In particular we will be interested in distinguishing its neoclassical and turbulent contributions. We begin from the Fokker-Planck equation given by

$$\frac{\partial f_i}{\partial t} + \mathbf{v}\cdot\nabla f_i + \frac{e}{m_i}\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right)\cdot\frac{\partial f_i}{\partial \mathbf{v}} = C. \tag{A1}$$

Taking the moment of Eq. (A1) with $m_i\mathbf{vv}$ yields

$$\frac{\partial}{\partial t}\mathbf{P}^{(i)} + \nabla\cdot\left(m_i\int d^3v f_i\mathbf{vvv}\right) + e\int d^3v\mathbf{vv}\frac{\partial}{\partial\mathbf{v}}\cdot(\mathbf{E}f_i) + \frac{e}{c}\int d^3v\mathbf{vv}(\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f_i}{\partial\mathbf{v}} = m_i\int d^3v\mathbf{vv}C, \tag{A2}$$

where $\mathbf{P}^{(i)}\equiv m_i\int d^3v f_i\mathbf{vv}$. This expression can be greatly simplified. The second term in Eq. (A2) can be expanded as (noting that after averaging only the radial component enters

$$\begin{aligned} m_i\int d^3v f_i(\hat{e}_r\cdot\mathbf{v})\mathbf{vv} = m_i&\int d^3v f_i[\hat{e}_r\cdot(\mathbf{v}-\mathbf{u})](\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u}) \\ &+ m_i\int d^3v f_i[\hat{e}_r\cdot(\mathbf{v}-\mathbf{u})](\mathbf{v}-\mathbf{u})\mathbf{u} \\ &+ m_i\mathbf{u}\int d^3v f_i[\hat{e}_r\cdot(\mathbf{v}-\mathbf{u})](\mathbf{v}-\mathbf{u}) \\ &+ m_i(\hat{e}_r\cdot\mathbf{u})\int d^3v f_i(\mathbf{v}-\mathbf{u})\mathbf{u} \\ &+ m_i(\hat{e}_r\cdot\mathbf{u})\int d^3v f_i(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u}) \\ &+ m_i\mathbf{u}(\hat{e}_r\cdot\mathbf{u})\int d^3v f_i(\mathbf{v}-\mathbf{u}) \\ &+ m_i n_i(\hat{e}_r\cdot\mathbf{u})\mathbf{uu}. \end{aligned} \tag{A3}$$

Neglecting third order terms in $\mathbf{u}$, approximating $m_i\int d^3v f_i(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u})\approx P_{i\perp}(\mathbf{I}-\hat{b}\hat{b})+P_{i\parallel}\hat{b}\hat{b}$, $m_i\int d^3v f_i[\hat{e}_r\cdot(\mathbf{v}-\mathbf{u})](\mathbf{v}-\mathbf{u})\approx\hat{e}_r P_{i\perp}$, and neglecting third order moments of $(\mathbf{v}-\mathbf{u})$ yields

$$m_i\int d^3v f_i(\hat{e}_r\cdot\mathbf{v})\mathbf{vv} = (\hat{e}_r\cdot\mathbf{u})[P_{i\perp}(\mathbf{I}-\hat{b}\hat{b}) + P_{i\parallel}\hat{b}\hat{b}] + (\mathbf{u}\hat{e}_r + \hat{e}_r\mathbf{u})P_{i\perp}. \tag{A4}$$

Similarly, the third and fourth terms in Eq. (A2) can be simplified as

$$-en_i\mathbf{Eu} - en_i\mathbf{uE} - \frac{e}{c}\int d^3v f_i\{(\mathbf{v}_\perp\times\mathbf{B})\mathbf{v} + \mathbf{v}(\mathbf{v}_\perp\times\mathbf{B})\}. \tag{A5}$$

From Eqs. (A4) and (A5), Eq. (A2) can be written (after averaging)

$$\frac{\partial\overline{\mathbf{P}^{(i)}}}{\partial t} + \mathbf{S} + \mathbf{T} - e\overline{\mathbf{E}n_i\mathbf{u}} - e\overline{n_i\mathbf{uE}} - \frac{e}{c}\int d^3v\overline{f_i\{(\mathbf{v}_\perp\times\mathbf{B})\mathbf{v} + \mathbf{v}(\mathbf{v}_\perp\times\mathbf{B})\}} = \mathbf{C}, \tag{A6}$$

where (in cylindrical coordinates)

$$S \equiv \frac{1}{r}\frac{\partial}{\partial r}\overline{ru_r[P_{i\perp}(\mathbf{I}-\hat{b}\hat{b}) + P_{i\parallel}\hat{b}\hat{b}]}, \tag{A7a}$$

$$T \equiv \frac{1}{r}\frac{\partial}{\partial r}\overline{rP_{i\perp}(\mathbf{u}\hat{e}_r + \hat{e}_r\mathbf{u})}, \tag{A7b}$$

$$C \equiv m_i\overline{\int d^3u\mathbf{vv}C}. \tag{A7c}$$

Operating on Eq. (A6) with $\hat{e}_y\hat{e}_y$ and averaging yields

$$\omega_{ci}\Pi_{ry}^{(i)} = e\overline{\delta E_y\delta(n_iu_y)} - \frac{1}{2}\left[\frac{\partial}{\partial t}\overline{P_{yy}^{(i)}} + T_{yy} + S_{yy} - C_{yy}\right]. \tag{A8a}$$

Similarly, for $\hat{e}_r\hat{e}_r$ we have

$$\omega_{ci}\Pi_{ry}^{(i)} = -e\overline{\delta E_r\delta(n_iu_r)} + \frac{1}{2}\left[\frac{\partial}{\partial t}\overline{P_{rr}^{(i)}} + T_{rr} + S_{rr} - C_{rr}\right]. \tag{A8b}$$

These expressions may be simplified by noting

$$\overline{P_{rr}^{(i)}} \approx \overline{P_{yy}^{(i)}} \approx m_i\overline{\int d^3v f_i v_y^2} \approx \bar{P}_{i\perp},$$

and $S_{rr}=S_{yy}$, such that summing Eqs. (A8a) and (A8b) yields

$$\omega_{ci}\Pi_{ry}^{(i)} = \tfrac{1}{2}e[\overline{\delta E_y\delta(n_iu_y)} - \overline{\delta E_r\delta(n_iu_r)}] + \tfrac{1}{4}[T_{rr} - T_{yy} + C_{yy} - C_{rr}]. \tag{A9}$$

In order to further simplify Eq. (A8b), it is useful to note that $T_{yy}=0$, and in the long wavelength limit it is straightforward to derive the relation:

$$e\overline{\delta E_y\delta(n_iu_y)} = -e\overline{\delta E_r\delta(n_iu_r)} + \tfrac{1}{2}T_{rr}, \tag{A10}$$

where we have estimated $\delta(n_iu_r)$ by the relation

$$\delta(n_iu_r) = n_0\delta u_r \approx -\frac{1}{m_i\omega_{ci}}\frac{\partial\delta P_{i\perp}}{\partial y} + n_0\frac{c}{B}\delta E_y,$$

and noted $\overline{u_r}=0$. Making these approximations, Eq. (A9) can be written as

$$\omega_{ci}\Pi_{ry}^{(i)} = e\overline{\delta E_y\delta(n_iu_y)} + \frac{1}{4}[C_{yy} - C_{rr}]. \tag{A11}$$

Similarly, in order to derive an expression for the parallel stress it is useful to operate on Eq. (A6) with $\hat{e}_y\hat{e}_\parallel$,

$$\omega_{ci}\Pi_{r\parallel}^{(i)} = e\{\overline{\delta E_\parallel \delta(n_i u_y)} + \overline{\delta E_y \delta(n_i u_\parallel)}\} - \left[\frac{\partial}{\partial t}\Pi_{y\parallel}^{(i)} + T_{y\parallel} + S_{y\parallel} - C_{y\parallel}\right]. \tag{A12}$$

Noting the inequality $\partial\Pi_{y\parallel}^{(i)}/\partial t \ll \omega_{ci}\Pi_{r\parallel}^{(i)}$, and that $T_{y\parallel}=S_{y_\parallel}=0$, Eq. (A12) can be written as

$$\omega_{ci}\Pi_{r\parallel}^{(i)} = e\{\overline{\delta E_\parallel \delta(n_i u_y)} + \overline{\delta E_y \delta(n_i u_\parallel)}\} + C_{y\parallel}. \tag{A13}$$

Thus, the turbulent contributions to the stress tensor are given by

$$\Pi_{ry}^{(\text{turb})} \equiv m_i \frac{c}{B}\overline{\delta E_y \delta(n_i u_y)}. \tag{A14a}$$

$$\Pi_{r\parallel}^{(\text{turb})} \equiv m_i \frac{c}{B}\{\overline{\delta E_\parallel \delta(n_i u_y)} + \overline{\delta E_y \delta(n_i u_\parallel)}\}, \tag{A14b}$$

where we have neglected the superscript i, since in the limit $m_e/m_i \to 0$, only the ions will contribute to the stress tensor.

APPENDIX B: DERIVATION OF TAYLOR IDENTITY FOR SLAB GEOMETRY

1. Transformation of perpendicular momentum to gyrocenter coordinates

In this subsection the perpendicular momentum given by

$$n_i u_y = \int d^3v v_y f_i \tag{B1}$$

will be transformed into gyrocenter coordinates. Following Ref. 40 closely, we will first transform the particle distribution function to guiding center coordinates, and then introduce a second transformation to gyrocenter coordinates. The first transformation can be written as

$$f_i = e^{-\rho_\perp\cdot\nabla_\perp} f_i', \tag{B2}$$

where f_i' is the guiding center distribution function and we have defined

$$\rho_\perp \equiv -\rho_\perp[\sin\alpha\hat{e}_r - \cos\alpha\hat{e}_y], \tag{B3a}$$

$$\mathbf{v}_\perp \equiv v_\perp[\cos\alpha\hat{e}_r + \sin\alpha\hat{e}_y]. \tag{B3b}$$

The gyrocenter distribution function can then be written as

$$f_i' = F_i + \{S_1, F_i\}, \tag{B4}$$

where

$$S_1 \equiv \frac{e}{m_i\omega_{ci}}\int^\alpha d\alpha(\phi_{gc} - \langle\phi_{gc}\rangle_\alpha), \tag{B5a}$$

$$\phi_{gc} = e^{\rho_\perp\cdot\nabla_\perp}\phi, \tag{B5b}$$

and the gyrocenter Poisson bracket can be written as

$$\{S_1,F_i\} = \frac{e}{m_i c}\left(\frac{\partial S_1}{\partial\alpha}\frac{\partial F_i}{\partial\mu} - \frac{\partial S_1}{\partial\mu}\frac{\partial F_i}{\partial\alpha}\right) + \frac{\mathbf{B}^*}{B^*}\cdot\left(\nabla S_1\frac{\partial F_i}{\partial v_\parallel} - \frac{\partial S_1}{\partial v_\parallel}\nabla F_i\right) - \frac{cm_i}{eB^*}\hat{b}\cdot(\nabla S_1\times\nabla F_i) + \left(\frac{\partial S_1}{\partial w}\frac{\partial F_i}{\partial t} - \frac{\partial S_1}{\partial t}\frac{\partial F_1}{\partial w}\right). \tag{B6}$$

Defining the quantities

$$\tilde{\phi} \equiv \phi_{gc} - \langle\phi_{gc}\rangle_\alpha, \tag{B7a}$$

$$\Phi \equiv \int^\alpha d\alpha\tilde{\phi}, \tag{B7b}$$

Eq. (B4) can be reduced to

$$\{S_1,F_i\} = \frac{e}{m_i B}\tilde{\phi}\frac{\partial F_i}{\partial\mu} + \frac{e}{m_i\omega_{ci}}\frac{\partial F_i}{\partial v_\parallel}\frac{\mathbf{B}^*}{B^*}\cdot\nabla\Phi - \frac{1}{\omega_{ci}}\frac{c}{B^*}\hat{b}\cdot(\nabla\Phi\times\nabla F_i). \tag{B8}$$

From Eqs. (B2), (B3b), (B4), and (B8), Eq. (B1) can be approximated as

$$n_i u_y = 2\pi\int d\mu dv_\parallel B^*\sqrt{2\mu B}\Bigg\langle \sin\alpha e^{-\rho_\perp\cdot\nabla_\perp} \times\left[F_i + \frac{e}{m_i B}\tilde{\phi}\frac{\partial\overline{F}_i}{\partial\mu} + \frac{c}{\omega_{ci}B^*}v_\parallel\frac{\partial\overline{F}_i}{\partial v_\parallel}(\nabla\times\hat{b})\cdot\nabla\Phi - \frac{c}{\omega_{ci}B^*}(\hat{b}\times\nabla\Phi)\cdot\nabla\overline{F}_i\right]\Bigg\rangle_\alpha. \tag{B9}$$

The third and fourth terms in brackets can be shown to be higher order in ρ_i/R and ρ_i/L_f, respectively, where $L_f^{-1} \equiv -d\ln\overline{F}_i/dr$. Thus, Eq. (B9) can be written to lowest nonvanishing order as

$$n_i u_y = 2\pi\int d\mu dv_\parallel B^*\sqrt{2\mu B}\Bigg\langle \sin\alpha e^{-\rho_\perp\cdot\nabla_\perp} \times\left[F_i + \frac{e}{m_i B}\tilde{\phi}\frac{\partial\overline{F}_i}{\partial\mu}\right]\Bigg\rangle_\alpha. \tag{B10}$$

After evaluating the integral

$$\langle\sin\alpha e^{-\rho_\perp\cdot\nabla_\perp}\rangle_\alpha = \frac{1}{2}[J_0(k_\perp\rho_\perp) + J_2(k_\perp\rho_\perp)]\rho_\perp\frac{\partial}{\partial r},$$

Eq. (B10) can be written as

$$n_i u_y = \frac{1}{\omega_{ci}}\int d^3\overline{v}\mu B[J_0(k_\perp\rho_\perp) + J_2(k_\perp\rho_\perp)]\frac{\partial}{\partial r} \times\left[F_i + J_0(k_\perp\rho_\perp)\frac{e\phi}{T_{\perp i}}\overline{F}_i\right]. \tag{B11}$$

In the long wavelength limit, the first and second terms in brackets can be identified as the diamagnetic and $E\times B$ components of the perpendicular momentum, respectively. This expression may be further simplified by noting

$$\int d\mu \mu B J_0^2(\lambda)\exp\left(-\frac{1}{2}\frac{v_\perp^2}{v_{\perp thi}^2}\right) = \frac{v_{\perp thi}^4}{B}\Gamma_0(b)\left[1-b\left(1-\frac{I_1(b)}{I_0(b)}\right)\right],$$

$$\int d\mu \mu B J_0(\lambda)J_2(\lambda)\exp\left(-\frac{1}{2}\frac{v_\perp^2}{v_{\perp thi}^2}\right) = \frac{v_{\perp thi}^4}{B}\Gamma_0(b)\left[b-(1+b)\frac{I_1(b)}{I_0(b)}\right].$$

Thus for a Maxwellian mean distribution, Eq. (B11) may be written as

$$n_i u_y = \frac{1}{\omega_{ci}}\int d^3\bar{v}\mu B[J_0(\lambda)+J_2(\lambda)]\frac{\partial F_i}{\partial r} - n_0\frac{v_{\perp thi}^2}{\omega_{ci}}[\Gamma_1(b)-\Gamma_0(b)]\frac{\partial}{\partial r}\left(\frac{e\phi}{T_{\perp i}}\right). \tag{B12}$$

2. Taylor identity in slab geometry

In this subsection a variant of the Taylor identity discussed above will be derived in a homogeneous slab geometry including finite Larmor radius corrections. The use of this idealized geometry will allow for a very general and transparent derivation of the Taylor identity without necessitating treatment of many of the subtle technical issues that arise in more complex geometries.

Similar to the derivation in Sec. III, the functions $J_\nu(\lambda)$ and $\Gamma_\nu(b)\equiv I_\nu(b)\exp(-b)$, where $\lambda\equiv k_\perp\rho_\perp$ and $b\equiv k_\perp^2\rho_i^2$, will appear frequently. Hence it is convenient to derive some properties of these functions. The series representation of J_0 can be written as

$$J_0(\lambda)=\sum_{m=0}^{\infty}\frac{(1/4)^m}{m!\Gamma(m+1)}|\rho_\perp\nabla_\perp|^{2m}, \tag{B13}$$

where Γ is a Gamma function, which can be distinguished from Γ_ν due to its lack of a subscript. Separating the perpendicular wavenumber into a slow and fast piece $\nabla_\perp\to\nabla_\perp^{(0)}+\varepsilon\nabla_\perp^{(1)}$, the lowest order contribution can be written as

$$J_0^{(0)}(\lambda)=\sum_{m=0}^{\infty}\frac{(1/4)^m}{m!\Gamma(m+1)}|\rho_\perp\nabla_\perp^{(0)}|^{2m}. \tag{B14}$$

To next order in ε, J_0 can be written as

$$J_0^{(1)}(\lambda)=-\rho_\perp^2\Xi_0[\nabla_\perp^{(1)}\cdot\nabla_\perp^{(0)}+\nabla_\perp^{(0)}\cdot\nabla_\perp^{(1)}], \tag{B15a}$$

$$\Xi_0\equiv\sum_{m=0}^{\infty}\frac{m2^{-2m}}{m!\Gamma(m+1)}|\rho_\perp\nabla_\perp^{(0)}|^{2(m-1)}. \tag{B15b}$$

The former expression can be written in a more transparent form by noting

$$\sum_{m=0}^{\infty}\frac{m2^{-2m}}{m!\Gamma(m+1)}|\rho_\perp\nabla_\perp^{(0)}|^{2(m-1)}=-\frac{1}{4}[J_0^{(0)}(\lambda)+J_2^{(0)}(\lambda)],$$

where $J_2^{(0)}$ is defined analogously to $J_0^{(0)}$. Thus, Eq. (B15) can be written as

$$J_0^{(1)}(\lambda)=\frac{1}{4}\rho_\perp^2[J_0^{(0)}(\lambda)+J_2^{(0)}(\lambda)][\nabla_\perp^{(1)}\cdot\nabla_\perp^{(0)}+\nabla_\perp^{(0)}\cdot\nabla_\perp^{(1)}]. \tag{B16}$$

Similarly, Γ_0 can be expanded,

$$\Gamma_0^{(0)}(b)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{2^{-2m}}{m!n!\Gamma(m+1)}|\rho_i\nabla_\perp^{(0)}|^{2(2m+n)}. \tag{B17}$$

To next order in ε,

$$\Gamma_0^{(1)}(b)=-\Lambda_0\rho_i^2[\nabla_\perp^{(1)}\cdot\nabla_\perp^{(0)}+\nabla_\perp^{(0)}\cdot\nabla_\perp^{(1)}], \tag{B18a}$$

$$\Lambda_0=-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2m+n)2^{-2m}}{m!n!\Gamma(m+1)}|\rho_i\nabla_\perp^{(0)}|^{2(2m+n-1)}. \tag{B18b}$$

This expression can be simplified by noting

$$-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2m+n)2^{-2m}}{m!n!\Gamma(m+1)}|\rho_i\nabla_\perp^{(0)}|^{2(2m+n-1)} = \Gamma_1^{(0)}(b)-\Gamma_0^{(0)}(b),$$

such that Eq. (B18) can be written as

$$\Gamma_0^{(1)}(b)=-\rho_i^2[\Gamma_1^{(0)}(b)-\Gamma_0^{(0)}(b)][\nabla_\perp^{(1)}\cdot\nabla_\perp^{(0)}+\nabla_\perp^{(0)}\cdot\nabla_\perp^{(1)}]. \tag{B19}$$

Utilizing these definitions, it is straightforward to derive some elementary properties of these operators. Namely for $J_{2\nu}^{(0)}$, where $\nu\equiv 0,1,2,\ldots$, it is easy to show that

$$\overline{\delta f J_{2\nu}^{(0)}(\lambda)\delta g}=\overline{\delta g J_{2\nu}^{(0)}(\lambda)\delta f}, \tag{B20}$$

i.e., the operation involves an even number of integration by parts and the surface terms vanish. Similarly, an analogous relation can be derived for $\Gamma_\nu^{(0)}$, i.e.,

$$\overline{\delta f \Gamma_\nu^{(0)}(b)\delta g}=\overline{\delta g \Gamma_\nu^{(0)}(b)\delta f}. \tag{B21}$$

Note that analogous relations *can not* be derived for $J_{2\nu}^{(1)}$ and $\Gamma_\nu^{(1)}$ due to the presence of surface terms.

The linearized quasineutrality relation can be expressed in gyrocenter coordinates as[38,39]

$$n_0\frac{e^2}{T_{\perp i}}[1-\Gamma_0(b)]\delta\phi+\rho_i^2\frac{e^2}{T_{\perp i}}\nabla_\perp n_0\cdot\nabla_\perp[\Gamma_1(b) - \Gamma_0(b)]\delta\phi=\sum_s q_s\int d^3\bar{v}J_0(\lambda)\delta F_s, \tag{B22}$$

where we have assumed the Debye length to be negligible compared to $k_\perp$ and have neglected nonlinearities. To lowest order in ε, Eqs. (B22) and (21) can be written as

$$\Upsilon^{(2)}=\sum_s q_s\overline{\int d^3\bar{v}\,\delta F_s^{(1)}(\hat{b}\times\nabla_\perp^{(0)}J_0^{(0)}\delta\phi^{(1)})_r}, \tag{B23a}$$

$$en_0[1-\Gamma_0^{(0)}]\frac{e\delta\phi^{(1)}}{T_i}=\sum_s q_s\int d^3\bar{v}J_0^{(0)}\delta F_s^{(1)}. \tag{B23b}$$

Utilizing Eq. (B20), Eq. (B23b) can be substituted into Eq. (B23a), yielding

$$\begin{aligned}
&-\overline{\sum_s q_s\int d^3\bar{v}\,\delta F_s^{(1)}\nabla_y^{(0)}J_0^{(0)}\delta\phi^{(1)}}\\
&=-\overline{en_0(1-\Gamma_0^{(0)})\frac{e\delta\phi^{(1)}}{T_i}\nabla_y^{(0)}\delta\phi^{(1)}}\\
&=-\frac{1}{2}n_0\frac{e^2}{T_i}\{\overline{(1-\Gamma_0^{(0)})\delta\phi^{(1)}\nabla_y^{(0)}\delta\phi^{(1)}}\\
&\quad+\overline{\delta\phi^{(1)}\nabla_y^{(0)}[(1-\Gamma_0^{(0)})\delta\phi^{(1)}]}\}\\
&=-\frac{1}{2}n_0\frac{e^2}{T_{\perp i}}\overline{\nabla_y^{(0)}\{[(1-\Gamma_0^{(0)})\delta\phi^{(1)}]\delta\phi^{(1)}\}}\\
&=0.
\end{aligned} \tag{B24}$$

Hence, to lowest order in ε the flux of gyrocenter charge vanishes.

To the next order, the approximate quasineutrality relation given by Eq. (B22) can be written as

$$\begin{aligned}
&-n_0\frac{e^2}{T_{\perp i}}\Gamma_0^{(1)}\delta\phi^{(1)}+n_0\frac{e^2}{T_{\perp i}}(1-\Gamma_0^{(0)})\delta\phi^{(2)}\\
&\quad+\rho_i^2\frac{e^2}{T_{\perp i}}\nabla_\perp^{(1)}n_0\cdot\nabla_\perp^{(0)}(\Gamma_1^{(0)}-\Gamma_0^{(0)})\delta\phi^{(1)}\\
&=\sum_s q_s\int d^3\bar{v}J_0^{(1)}\delta F_s^{(1)}+\sum_s q_s\int d^3\bar{v}J_0^{(0)}\delta F_s^{(2)}.
\end{aligned} \tag{B25a}$$

Similarly, the next order contribution from the gyrocenter flux can be written as

$$\begin{aligned}
\Upsilon^{(3)}&=\overline{\sum_s q_s\int d^3\bar{v}\,\delta F_s^{(1)}(\hat{b}\times\nabla_\perp^{(0)}J_0^{(0)}\delta\phi^{(2)})_x}\\
&\quad+\overline{\sum_s q_s\int d^3\bar{v}\,\delta F_s^{(1)}(\hat{b}\times\nabla_\perp^{(0)}J_0^{(1)}\delta\phi^{(1)})_x}\\
&\quad+\overline{\sum_s q_s\int d^3\bar{v}\,\delta F_s^{(1)}(\hat{b}\times\nabla_\perp^{(1)}J_0^{(0)}\delta\phi^{(1)})_x}\\
&\quad+\overline{\sum_s q_s\int d^3\bar{v}\,\delta F_s^{(2)}(\hat{b}\times\nabla_\perp^{(0)}J_0^{(0)}\delta\phi^{(1)})_x}.
\end{aligned} \tag{B25b}$$

$\Upsilon^{(3)}$ may be simplified via an analogous procedure as that utilized in the main body of the text, yielding

$$\begin{aligned}
\Upsilon&=e\frac{\partial}{\partial x}\overline{\left\{n_0\delta E_y(\Gamma_1-\Gamma_0)\frac{\partial}{\partial x}\left(\rho_i^2\frac{e\delta\phi}{T_i}\right)\right\}}\\
&\quad-\frac{1}{2}\frac{\partial}{\partial x}\overline{\left\{\delta E_y\sum_s q_s\int d^3\bar{v}\rho_\perp^2(J_0+J_2)\frac{\partial}{\partial x}\delta F_s\right\}},
\end{aligned} \tag{B26}$$

where we have dropped all superscripts in order to simplify the notation. Substituting the perpendicular momentum written in gyrocenter coordinates [Eq. (B11)] into Eq. (B26) for $m_e/m_i\to 0$ yields

$$\begin{aligned}
\Upsilon&\equiv\overline{\sum_s q_s\int d^3\bar{v}\,\delta F_s(\hat{b}\times\nabla_\perp J_0\delta\phi)_x}\\
&=-m_i\frac{\partial}{\partial x}\left[\frac{c}{B}\overline{\delta(n_iu_y)\delta E_y}\right],
\end{aligned} \tag{B27}$$

thus providing a kinetic generalization of the Taylor identity given by Eq. (2) above.

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